Övningar till

ANALYS I FLERA VARIABLER

LTH 1996

Lösningar

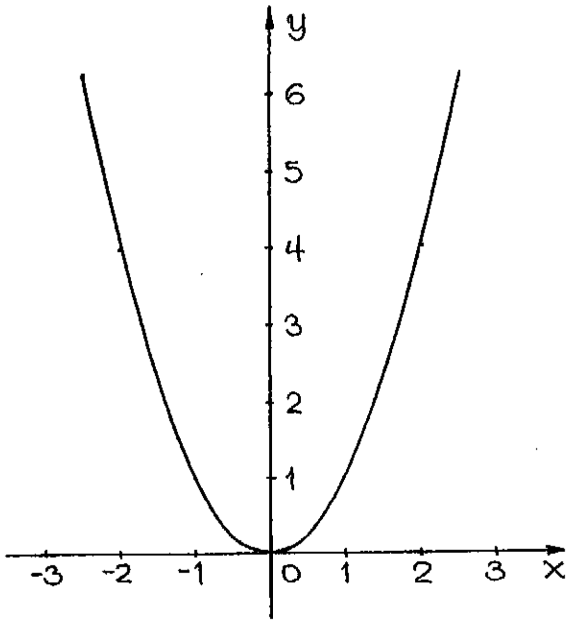
Vasilios Agapios

1. Funktioner av flera variabler
Rummet $\mathbb{R}^n$ och mängder i $\mathbb{R}^n$

Övning 1.1 (s. 1)

a) $y = x^2$

x	0	±0,5	±1	±1,5	±2	±2,5	±3
y	0	0,25	1	2,25	4	6,25	9



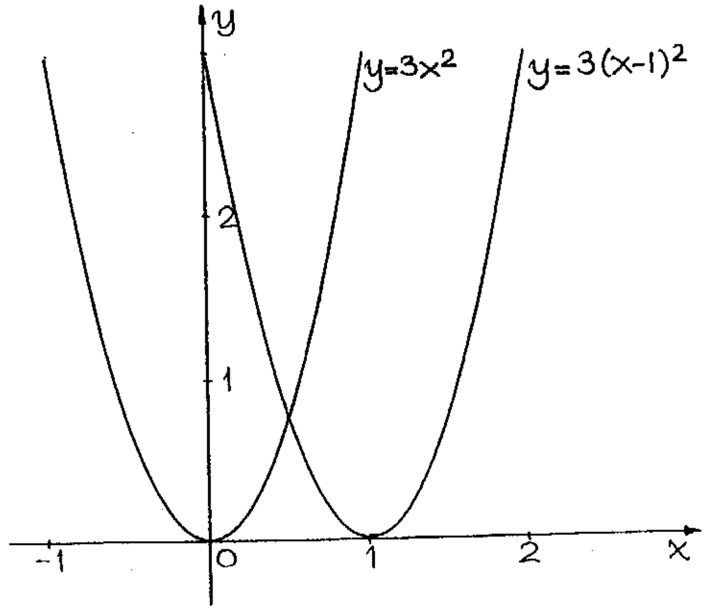
Ovanstående kurva är välkänd redan i gymnasiet. Mer om parabler kan du finna i BETA och på s. 82 (Third Edition.)

b) $y = 3x^2$

x	0	±0,2	±0,4	±0,6	±0,8	±1
y	0	0,12	0,48	1,08	1,92	3

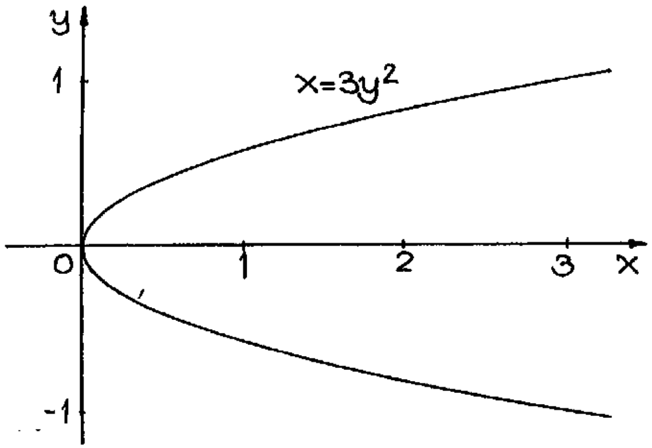
c) $y = 3(x-1)^2$

Man får kurvan $y = 3(x-1)^2$ genom att förskjuta kurvan $y = 3x^2$ 1 längdenhet åt höger (se fig.).



d) $3y^2 = x$

Denna kurva fås genom en vridning av kurvan $y = 3x^2$ 90° medurs kring origo.



<u>Anm.</u> Parabeln studeras även i linjär algebra i samband med diagonalisering av kvadratiska former.

Övning 1.2 (s.1)

a) $\frac{x^2}{4}+\frac{y^2}{9}=1$

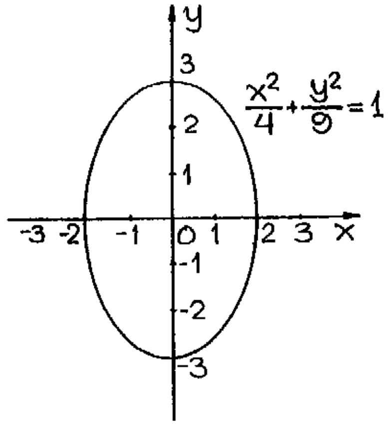
Om ellipsen kan du läsa i BETA på s. 81.

$$\frac{x^2}{4}+\frac{y^2}{9}=1 \Leftrightarrow \frac{x^2}{2^2}+\frac{y^2}{3^2}=1 \Rightarrow \begin{cases} x=2\cos t \\ y=3\sin t \\ 0\leq t\leq 2\pi \end{cases} ;$$

t	0	π/6	π/4	π/3	π/2	2π/3	3π/4	5π/6	π	7π/6
x	2	1,73	1,41	1	0	−1	−1,41	−1,73	−2	−1,73
y	0	1,50	2,12	2,60	3	2,60	2,12	1,50	0	−1,50

5π/4	4π/3	3π/2	5π/3	7π/4	11π/6	2π
−1,41	−1	0	1	1,41	1,73	2
−2,12	−2,60	−3	−2,60	−2,12	−1,50	0

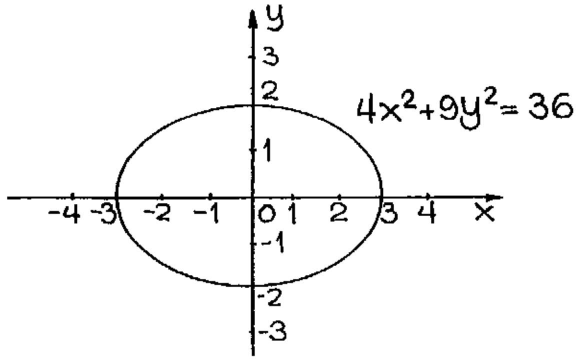
forts.



b) $4x^2+9y^2=36$

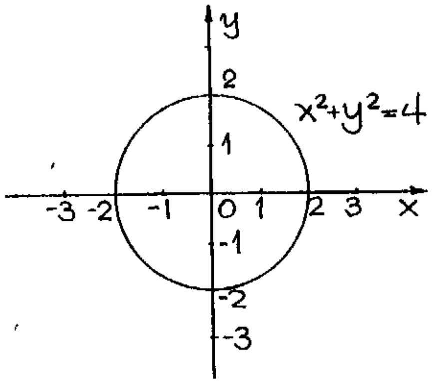
$$4x^2+9y^2=36 \Leftrightarrow \frac{4x^2}{36}+\frac{9y^2}{36}=1 \Leftrightarrow \frac{x^2}{3^2}+\frac{y^2}{2^2}=1.$$

Denna ellips får vi om vi i ellipsen ovan låter x och y byta plats. Geometriskt motsvarar detta en vridning kring origo 90° medurs.



c) $x^2+y^2=4$

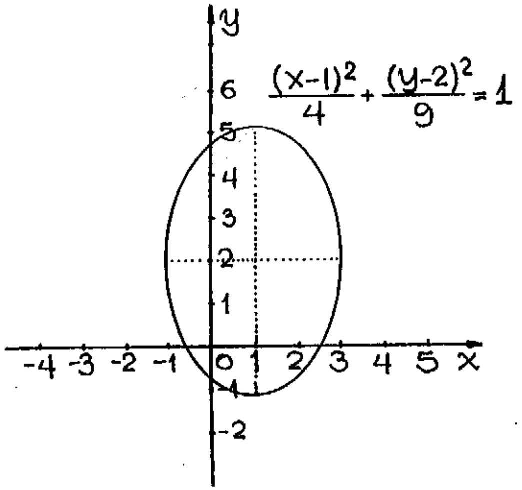
$x^2+y^2=4=2^2$; cirkel med centrum i origo och radien 2 (se fig. nedan.)



Anm. Om cirkeln kan du läsa i BETA (s.81)

d) $\dfrac{(x-1)^2}{4}+\dfrac{(y-2)^2}{9}=1.$

Samma ellips som i a) ovan men med centrum förlagt i punkten (1,2) (se fig.).



Övning 1.3 (s.1)

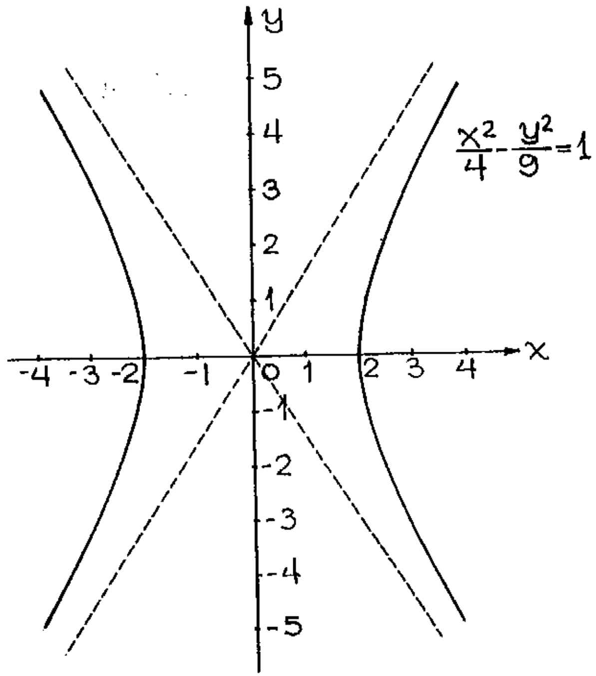
a) $\dfrac{x^2}{4}-\dfrac{y^2}{9}=1.$

$\dfrac{x^2}{4}-\dfrac{y^2}{9}=1 \Leftrightarrow \dfrac{x^2}{4}=\dfrac{y^2}{9}+1 \Leftrightarrow x=4\left(\dfrac{y^2}{9}+1\right) \Leftrightarrow x=\pm 2\sqrt{\dfrac{y^2}{9}+1}.$

y	0	±0,5	±1	±1,5	±2	±2,5	±3	±3,5	±4
x	±2	±2,03	±2,11	±2,24	±2,40	±2,60	2,82	±3,07	±3,33

Asymptoterna får vi ur ekvationen $\dfrac{x^2}{4}-\dfrac{y^2}{9}=0.$

Dessa är $y=\pm\frac{3}{2}x.$

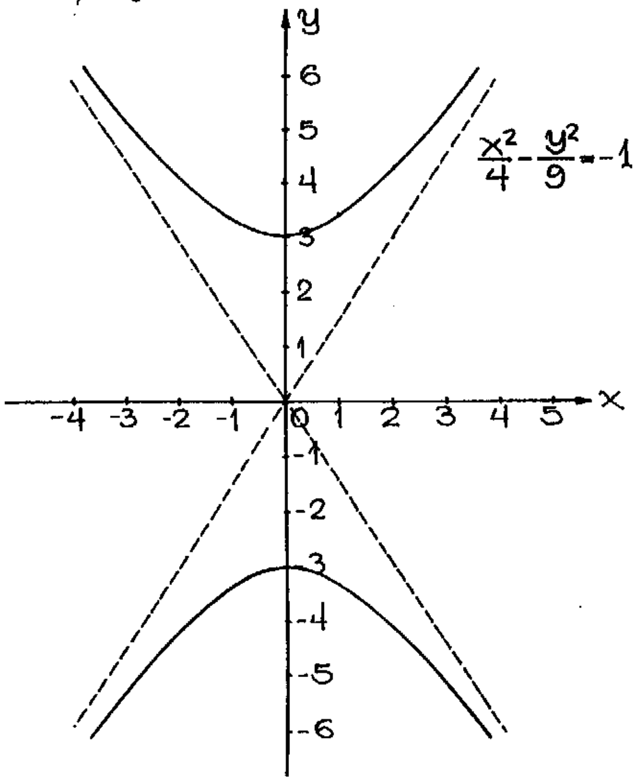


Anm. Om hyperbeln kan du läsa i BETA och på sidan 82.

b) $\frac{x^2}{4}-\frac{y^2}{9}=-1 \Leftrightarrow \frac{y^2}{9}=\frac{x^2}{4}+1 \Leftrightarrow y^2=9(\frac{x^2}{4}+1) \Leftrightarrow y=\pm 3\sqrt{\frac{x^2}{4}+1}$.

x	0	±0,5	±1	±1,5	±2	±2,5	±3	±3,5
y	±3	±3,09	±3,35	±3,75	±4,24	±4,80	±5,41	±6,05

Asymptoterna fås ur ekvationen $\frac{x^2}{4}-\frac{y^2}{9}=0$ och är identiska med asymptoterna i den föregående deluppgiften.



Anm. Ovanstående hyperbel är den konjugerade hyperbeln till föregående.

Övning 1.4 (s.1)

a) Avståndsformeln finns i BETA på s. 79.

$|(x,y)-(1,-2)| = |(x-1,y+2)| = \sqrt{(x-1)^2+(y+2)^2}$;

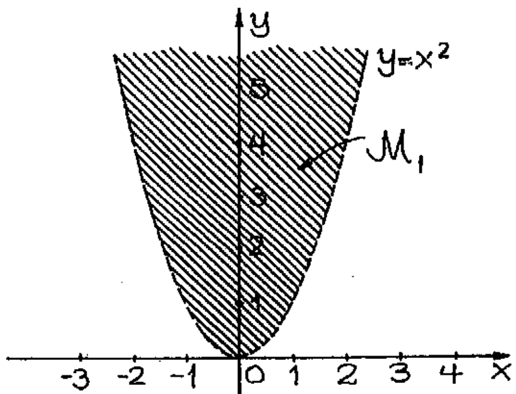
b) $\sqrt{(x-1)^2+(y+2)^2}=3 \Leftrightarrow (x-1)^2+(y+2)^2=3^2=9 \Leftrightarrow$

$\Leftrightarrow x^2-2x+1+y^2+4y+4=9 \Leftrightarrow x^2+y^2-2x+4y-5=0.$

Övning 1.5 (s.1)

a) $M_1=\{(x,y): y>x^2\}$.

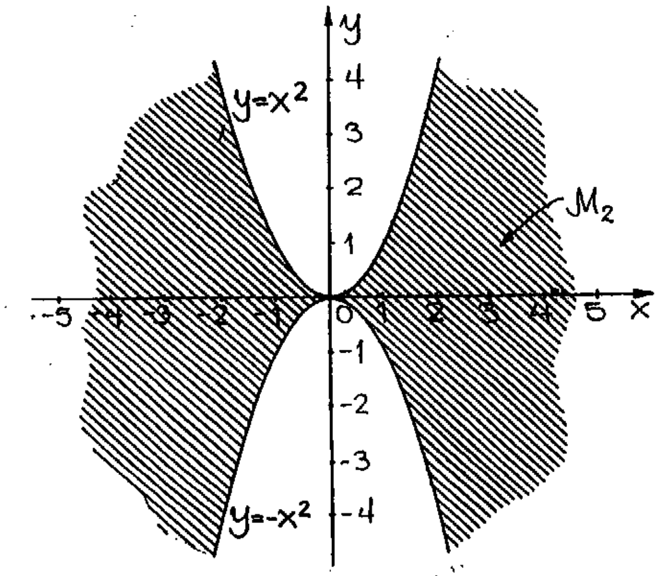
Kurvan $y=x^2$ delar xy-planet i två områden: $y>x^2$ och $y<x^2$. Vilket område representeras av M_1? För att ta reda på det, tar vi en punkt som satisfierar $y>x^2$. (0,1) är en sådan punkt. Vi tar och skuggar hela det område punkten ligger i.



b) $M_2 = \{(x,y) : |y| \le x^2\}$.

$|y| \le x^2 \Leftrightarrow -x^2 \le y \le x^2 \Leftrightarrow y \ge -x^2 \wedge y \le x^2$.

M_2 består av de punkter som ligger under och på parabeln $y = x^2$ och över och på parabeln $y = -x^2$ (Se fig.).



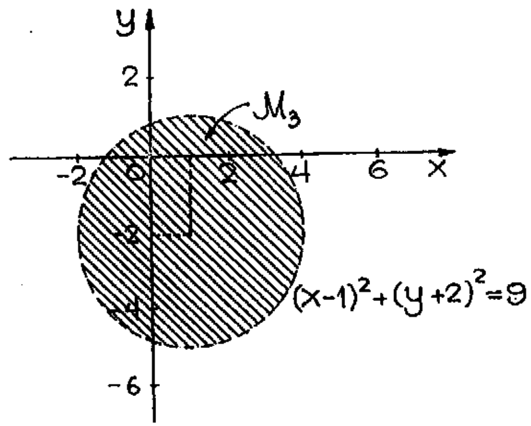
c) $M_3 = \{(x,y) : x^2 - 2x + y^2 + 4y < 4\}$.

$x^2 - 2x + y^2 + 4y = (x-1)^2 + (y+2)^2 - 5$;

$x^2 - 2x + y^2 + 4y < 4 \Leftrightarrow (x-1)^2 + (y+2)^2 - 5 < 4 \Leftrightarrow$

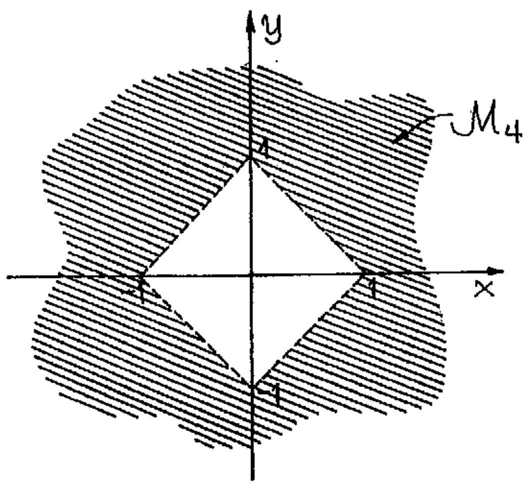
$\Leftrightarrow (x-1)^2 + (y-2)^2 < 9 \Leftrightarrow \sqrt{(x-1)^2 + (y+2)^2} < 3$.

M_3 består av de punkter som ligger innanför cirkeln med centrum i $(1,-2)$ och radien 3.



d) $M_4 = \{(x,y) : |y| > 1 - |x|\}$.

$|y| > 1 - |x| \Leftrightarrow \pm y > 1 - |x| \Leftrightarrow y > 1 - |x| \vee y < |x| - 1$.



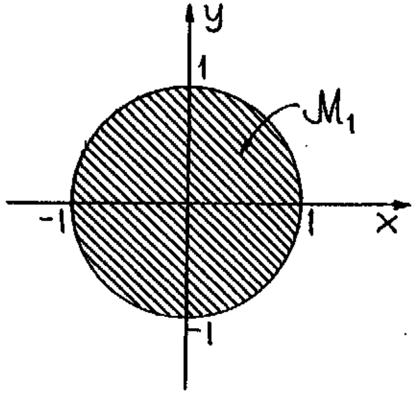
M_4 består av de punkter som ligger över kurvan $y=1-|x|$ och under kurvan $y=|x|-1$.

Anm. $|x|+|y|=1$ är ekvationen för en kvadrat med hörnen i $(\pm1,0)$ och $(0,\pm1)$. (Se nedan.)

Övning 1.6 (S.1)

a) $M_1=\{(x,y): x^2+y^2\le 1\}$.

M_1 består av punkterna innanför och på cirkeln $x^2+y^2=1$.



Anm. M_1 går under namnet enhetsdisken.

b) $M_2=\{(x,y): |x|+|y|<1\}$.

Låt oss analysera ordentligt vilken är innebörden av olikheten $|x|+|y|<1$!

forts.

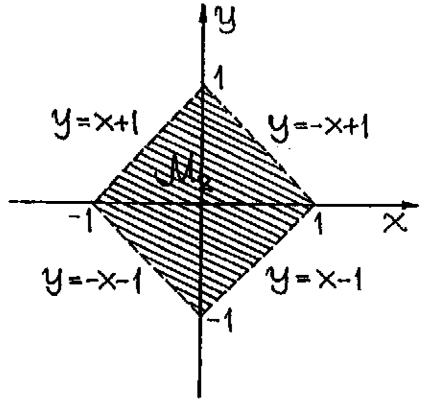
(i) $\begin{cases} x\ge 0 \Rightarrow |x|=x \\ y\ge 0 \Rightarrow |y|=y \end{cases} \Rightarrow |x|+|y|=x+y<1 \Leftrightarrow y<1-x$.

(ii) $\begin{cases} x<0 \Rightarrow |x|=-x \\ y\ge 0 \Rightarrow |y|=y \end{cases} \Rightarrow |x|+|y|=-x+y<1 \Leftrightarrow y<1+x$.

(iii) $\begin{cases} x<0 \Rightarrow |x|=-x \\ y<0 \Rightarrow |y|=-y \end{cases} \Rightarrow |x|+|y|=-x-y<1 \Leftrightarrow y>-1-x$.

(iv) $\begin{cases} x\ge 0 \Rightarrow |x|=x \\ y<0 \Rightarrow |y|=-y \end{cases} \Rightarrow |x|+|y|=x-y<1 \Leftrightarrow y>x-1$.

M_2 består alltså av 4 trianglar, en i varje kvadrant, dvs. en kvadrat som i figuren.

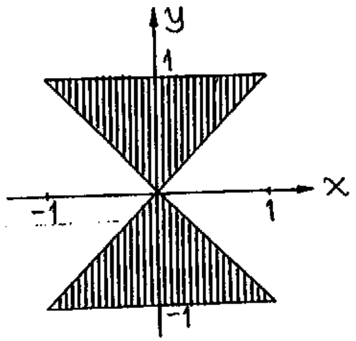


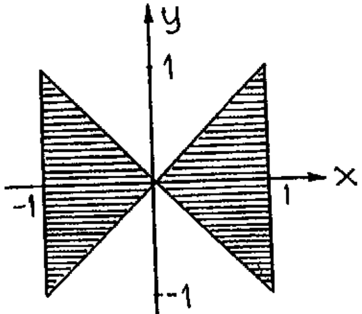
c) $M_3=\{(x,y): \max\{|x|,|y|\}\le 1\}$.

$|x|<|y| \Rightarrow \max\{|x|,|y|\}=|y|\le 1 \Leftrightarrow -1\le y\le 1$;

$|x|>|y| \Rightarrow \max\{|x|,|y|\} = |x| \leq 1 \Leftrightarrow -1 \leq x \leq 1.$

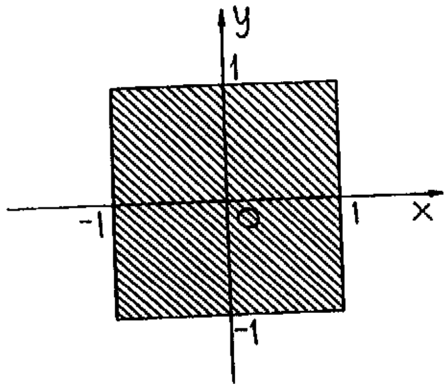
$\max\{\ \}$ utläses "det största av".





$|x|<|y|\leq 1$, $\qquad |y|<|x|\leq 1$

$\max\{|x|,|y|\} \leq 1 \Leftrightarrow \underline{|x|\leq 1 \wedge |y|\leq 1}$ (se nedan).



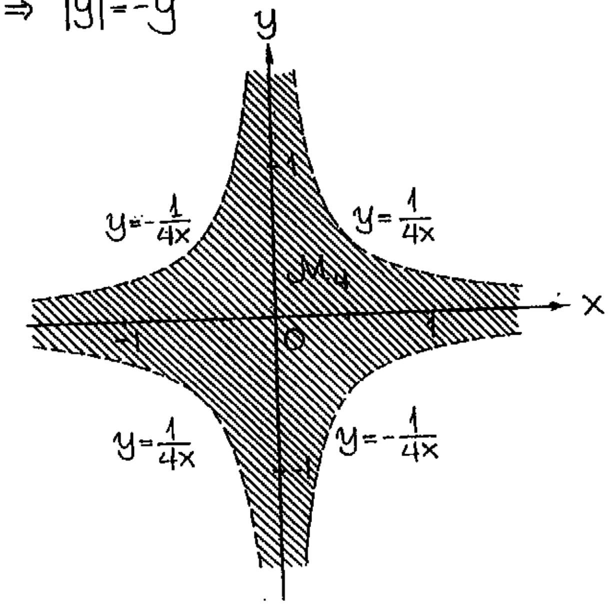
d) $\underline{M_4 = \{(x,y) :\ |xy| < \frac{1}{4}\}}.$ $\qquad (\,|x\cdot y| = |x|\cdot|y|\,)$

(i) $\begin{cases} \underline{x\geq 0} \Rightarrow |x| = x \\ \underline{y\geq 0} \Rightarrow |y| = y \end{cases} \Rightarrow |xy| = xy < \frac{1}{4} \Leftrightarrow \underline{y < \frac{1}{4}\cdot\frac{1}{x}}.$

(ii) $\begin{cases} \underline{x<0} \Rightarrow |x| = -x \\ \underline{y\geq 0} \Rightarrow |y| = y \end{cases} \Rightarrow |xy| = -xy < \frac{1}{4} \Leftrightarrow y < -\frac{1}{4}\frac{1}{x};$

(iii) $\begin{cases} \underline{x<0} \Rightarrow |x| = -x \\ \underline{y<0} \Rightarrow |y| = -y \end{cases} \Rightarrow |xy| = xy < \frac{1}{4} \Leftrightarrow y > \frac{1}{4}\frac{1}{x};$

(iv) $\begin{cases} \underline{x\geq 0} \Rightarrow |x| = x \\ \underline{y<0} \Rightarrow |y| = -y \end{cases} \Rightarrow |xy| = -xy < \frac{1}{4} \Leftrightarrow y > -\frac{1}{4}\frac{1}{x};$



e) $\underline{M_5 = \{(x,y) :\ x^2+y^2-2x-4y < 11\}}.$

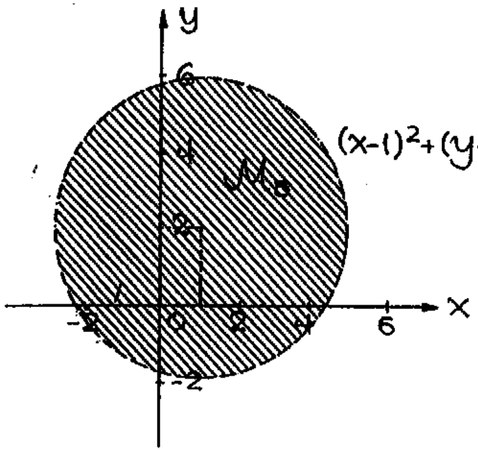
$x^2+y^2-2x-4y \equiv (x^2-2x+1)+(y^2-4y+4)-5 \equiv$

$\equiv (x-1)^2+(y-2)^2-5\ ;$

$x^2+y^2-2x-4y < 11 \Leftrightarrow (x-1)^2+(y-2)-5 < 11 \Leftrightarrow$

$$\Leftrightarrow (x-1)^2+(y-2)^2<16 \Leftrightarrow \sqrt{(x-1)^2+(y-2)^2}<4$$

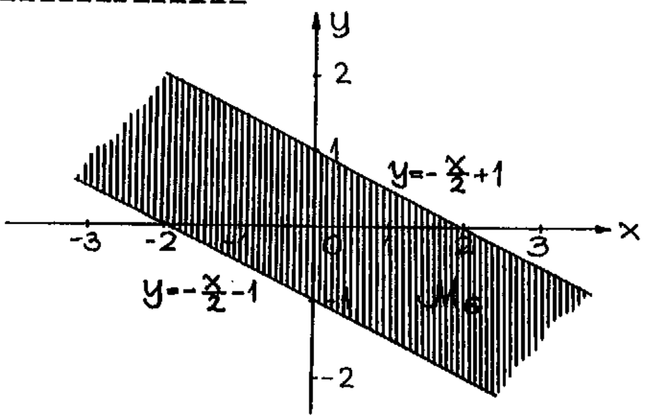
M_5 föreställer det inre av en cirkel med centrum i $(1,2)$ och radien 4 (se fig. nedan).



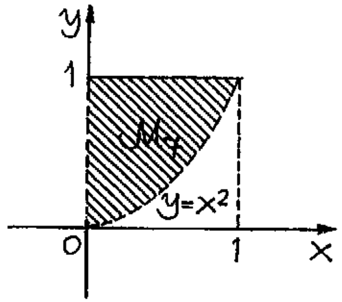
f) $\underline{M_6=\{(x,y):\ |x+2y|\leq 2\}}$.

$$|x+2y|\leq 2 \Leftrightarrow -2\leq x+2y\leq 2 \Leftrightarrow -2-x\leq 2y\leq -x+2 \Leftrightarrow$$

$$\Leftrightarrow \underline{-\tfrac{1}{2}x-1\leq y\leq -\tfrac{1}{2}x+1}\,;$$



g) $\underline{M_7=\{(x,y):\ y>x^2,\ 0<x<1,\ y\leq 1\}}$.

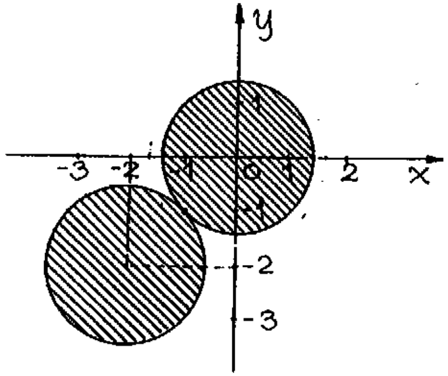


h) $\underline{M_8=\{(x,y):\ x^2+y^2\leq 2\leq -4-4x-4y-x^2-y^2\}}$.

Anm. $a\leq b\leq c \Leftrightarrow a\leq b \wedge b\leq c$.

$$\begin{cases} x^2+y^2\leq 2 \\ 2\leq -4-4x-4y-x^2-y^2 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2\leq 2 \\ x^2+4x+4+y^2+4y+4\leq 2 \end{cases}$$

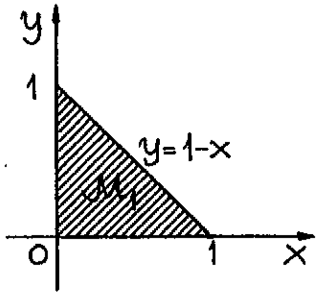
$$\Leftrightarrow \begin{cases} x^2+y^2\leq 2 \\ (x+2)^2+(y+2)^2\leq 2 \end{cases} \Leftrightarrow \begin{cases} \sqrt{x^2+y^2}\leq \sqrt{2}. \\ \sqrt{(x+2)^2+(y+2)^2}\leq \sqrt{2}. \end{cases}$$



$M_8=\{(-1,-1)\}$, ty $=|(-2,-2)-(0,0)|=2\cdot\sqrt{2}=2$ radier.

Övning 1.7 (s. 2)

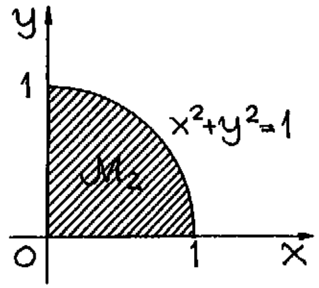
a)



$M_1 = \{(x,y):\ 0 \le y \le 1-x,\ x \ge 0\}.$

Anm. M_1 är mängden av alla punkter i den första kvadranten som ligger under och på linjen $y=1-x$. Axelsidorna ingår i M_1.

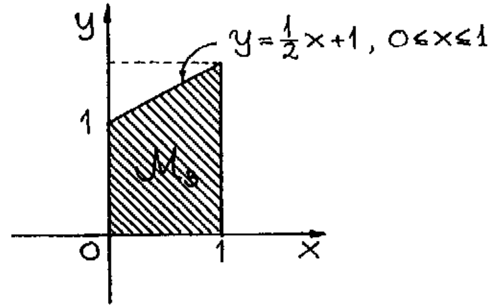
b)



$M_2 = \{(x,y):\ y \le \sqrt{1-x^2},\ 0 \le x \le 1\}.$

Anm. M_2 är mängden av alla punkter i den första kvadranten som ligger innanför och på enhetscirkeln. Axelsidorna ingår.

c)

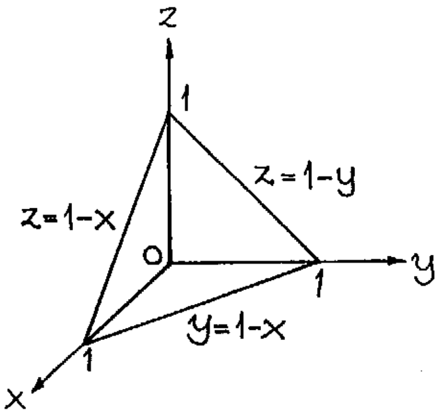


$M_3 = \{(x,y):\ 0 \le y \le \frac{1}{2}x+1,\ 0 \le x \le 1\}$

Anm. M_3 är mängden av alla punkter i den första kvadranten som ligger under och på linjen $y=\frac{1}{2}x+1$ och på bandet $0 \le x \le 1$. Axelsidorna ingår i M_3.

Övning 1.8 (s. 2)

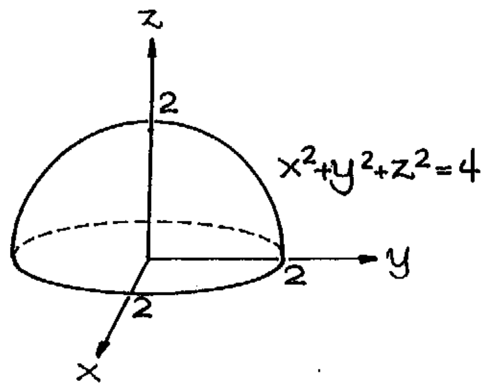
a) $M_1 = \{(x,y,z):\ x+y+z \le 1,\ x \ge 0,\ y \ge 0,\ z \ge 0\}.$



forts.

M_1 består av de punkter i den första oktanten som ligger på och under planet $x+y+z=1$. De sidor som sammanfaller med xy-, yz- och xz-planen ingår i mängden. M_1 representerar en tetraeder.

b) $M_2 = \{(x,y,z) : x^2+y^2+z^2<4,\ z>0\}$.

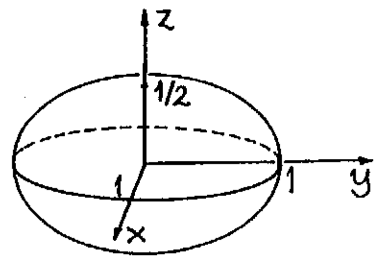


M_2 består av de punkterna i övre halvrummet som ligger under sfären $|\mathbf{x}|=2$.

Anm. Om andragradsytorna kan du läsa i BETA på s. 85-86.

c) $M_3 = \{(x,y,z) : x^2+y^2+4z^2 \le 1\}$.

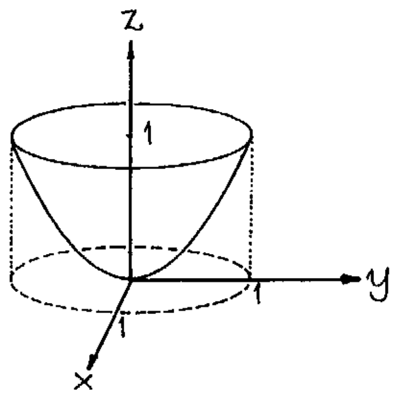
$$x^2+y^2+4z^2=1 \Leftrightarrow \frac{x^2}{1^2}+\frac{y^2}{1^2}+\frac{z^2}{(1/2)^2}=1.$$



M_3 består av alla punkter innanför och på (rotations)ellipsoiden med medelpunkten i origo och halvaxlarna 1, 1 resp. $\frac{1}{2}$.

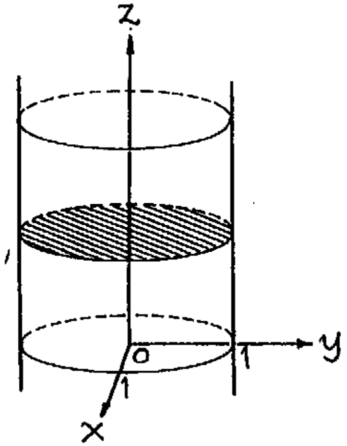
d) $M_4 = \{(x,y,z) : x^2+y^2 \le z \le 1\}$.

$$x^2+y^2 \le z \le 1 \Leftrightarrow x^2+y^2 \le z \ \wedge\ z \le 1.$$



M_4 består av alla punkter ovanför och på (rotations)paraboloiden $z=x^2+y^2$ och under och på cirkeln $x^2+y^2 \leq 1$, $z=1$.

e) $M_5 = \{(x,y,z) : x^2+y^2 \leq 1\}$



M_5 består av alla punkter innanför och på en oändligt lång cirkulär cylinder med radien 1 och (huvud)axeln (längs) z-axeln.

Övning 1.9 (s. 2)

a) $M_1 = \{(x,y) : x^2+y^2 \leq 1\} \Rightarrow \partial M_1 = \{(x,y) : x^2+y^2 = 1\}$.

b) $M_2 = \{(x,y) : |x|+|y| < 1\} \Rightarrow \partial M_2 = \{(x,y) : |x|+|y| = 1\}$.

c) $M_3 = \{(x,y) : \max\{|x|,|y|\} \leq 1\} \Rightarrow$

forts.

$\Rightarrow \partial M_3 = \{(x,y) : \max\{|x|,|y|\} = 1\}$.

d) $M_4 = \{(x,y) : |xy| < \frac{1}{4}\} \Rightarrow \partial M_4 = \{(x,y) : |xy| = \frac{1}{4}\}$.

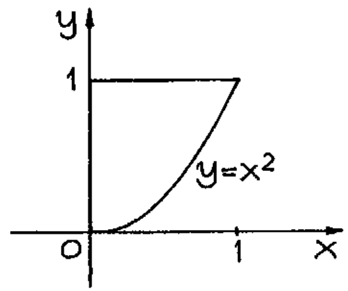
e) $M_5 = \{(x,y) : (x-1)^2+(y-2)^2 < 16\}$;

$\partial M_5 = \{(x,y) : (x-1)^2+(y-2)^2 = 16\}$.

f) $M_6 = \{(x,y) : |x+2y| \leq 2\} = \{(x,y) : -2 \leq x+2y \leq 2\}$;

$\partial M_6 = \{(x,y) : x+2y=-2\} \cup \{(x,y) : x+2y=2\}$.

g) $M_7 = \{(x,y) : y > x^2,\ 0 < x < 1,\ y \leq 1\}$.



$\partial M_7 = \{(x,y) : y=x^2,\ 0 \leq x \leq 1\} \cup \{(x,1) : 0 \leq x \leq 1\} \cup$
$\cup \{(0,y) : 0 \leq y \leq 1\}$.

h) $M_8 = \{(-1,-1)\} = \partial M_8$.

Övning 1.10 (s. 2)

Anm. $\mathring{M} = \{\text{inre punkter till } M\} = M \setminus \partial M$.

a) $M_1 = \{(x,y) : x^2+y^2 \leq 1\} \Rightarrow \mathring{M}_1 = \{(x,y) : x^2+y^2 < 1\}$.

b) $M_2 = \{(x,y) : |x|+|y|<1\} = \mathring{M}_2.$

c) $M_3 = \{(x,y) : \max\{|x|,|y|\} \leq 1\}$;

$\mathring{M}_3 = \{(x,y) : \max\{|x|,|y|<1\}.$

d) $M_4 = \{(x,y) : |xy| < \frac{1}{4}\} = \mathring{M}_4.$

e) $M_5 = \{(x,y) : x^2+y^2-2x-4y<11\} = \mathring{M}_5.$

f) $M_7 = \{(x,y) : y>x^2,\ 0<x<1,\ y\leq 1\}$;

$\mathring{M}_7 = \{(x,y) : y>x^2,\ 0<x<1,\ y<1\}.$

g) $M_8 = \{(-1,-1)\} \Rightarrow \mathring{M} = \emptyset.$

Övning 1.11 (s. 2)

Anm. En mängd M är öppen om $M = \mathring{M}$.

a) Mängderna M_2, M_4 och M_5 är öppna.

Anm. En mängd M är sluten om $\partial M \subseteq M$.

b) Mängderna M_1, M_3, M_6 och M_8 är slutna.

c) Mängden M_7 är varken öppen eller sluten.

Anm. En mängd M (i planet) är begränsad om den kan instängas i en cirkel med ändlig radie (!).

forts.

d) M_1 är begränsad, ty $M_1 \subseteq \{(x,y) : x^2+y^2 \leq 2\}.$

M_2 är begränsad, ty $M_2 \subseteq \{(x,y) : x^2+y^2 \leq 4\}.$

M_3 är begränsad, ty $M_3 \subseteq \{(x,y) : x^2+y^2 \leq 4\}.$

M_5 är begränsad, ty $M_5 \subseteq \{(x,y) : x^2+y^2 \leq 100\}.$

M_7 är begränsad, ty $M_7 \subseteq \{(x,y) : x^2+y^2 \leq 4\}.$

M_8 är begränsad, ty $(-1,-1) \in \{(x,y) : x^2+y^2 \leq 4\}.$

Anm: En mängd är kompakt om den är både sluten och begränsad.

e) Mängderna M_1, M_3 och M_8 är kompakta.

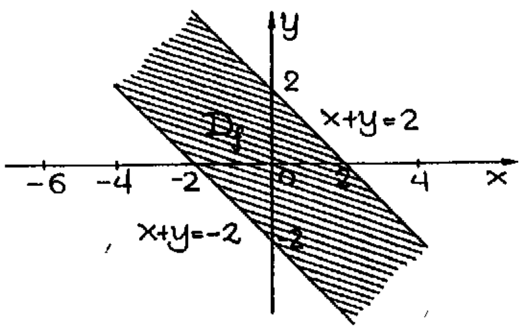
Funktioner från $\mathbb{R}^n$ till $\mathbb{R}^p$

Övning 1.12 (s. 2)

a) $f(x,y) = \sqrt{4-x^2-2xy-y^2}.$

$f(x,y) = \sqrt{\phi(x,y)}\ ,\ \phi(x,y) = 4-(x+y)^2.$

$$D_f = \{(x,y)\in D_\phi : \phi(x,y) \in D_{\sqrt{\ }}\} =$$
$$= \{(x,y)\in\mathbb{R}^2 : 4-(x+y)^2 \geq 0\} =$$
$$= \{(x,y)\in\mathbb{R}^2 : (x+y)^2 \leq 4\} =$$
$$= \{(x,y)\in\mathbb{R}^2 : -2 \leq x+y \leq 2\} \quad \text{(Se fig.)}$$

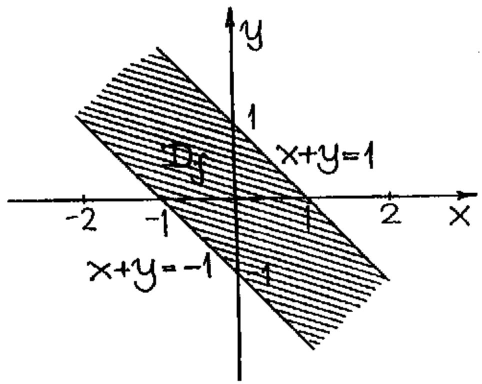


b) $f(x,y)=\arcsin(x+y)$.

$f(x,y)=\arcsin\phi(x,y)$, $\phi(x,y)=x+y$;

$D_f=\{(x,y)\in D_\phi;\ \phi(x,y)\in D_{\arcsin}\}=$

$=\{(x,y)\in\mathbb{R}^2:\ -1\leq x+y\leq 1\}$. (Se fig. nedan.)



c) $f(x,y)=\ln\frac{x+y}{x-y}$.

$f(x,y)=\ln\phi(x,y)$, $\phi(x,y)=\frac{x+y}{x-y}$;

$D_f=\{(x,y)\in D_\phi:\ \phi(x,y)\in D_{\ln}\}=$

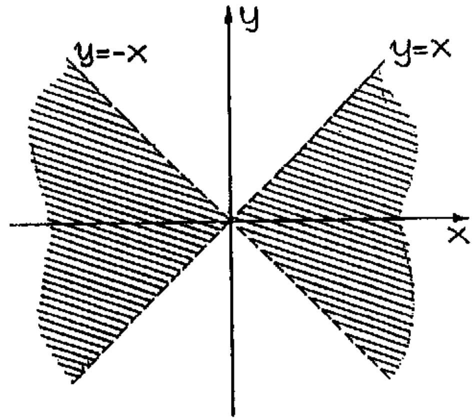
$=\{(x,y)\in\complement\{(t,t):t\in\mathbb{R}\}:\ \frac{x+y}{x-y}>0\}=$

$=\{(x,y)\in\mathbb{R}^2:\ (x+y)(x-y)>0\}=$

$=\{(x,y)\in\mathbb{R}^2:\ x^2-y^2>0\}=\{(x,y):\ y^2<x^2\}=$

$=\{(x,y):\ |y|<|x|\}=\underline{\{(x,y):\ -|x|<y<|x|\}}$.

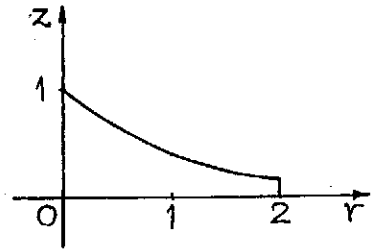
Anm. $y^2<x^2\Leftrightarrow\sqrt{y^2}<\sqrt{x^2}\Leftrightarrow|y|<|x|$ (Se fig.)



Övning 1.13 (s. 2)

a) $z=e^{-r}$, $0\leq r\leq 2$

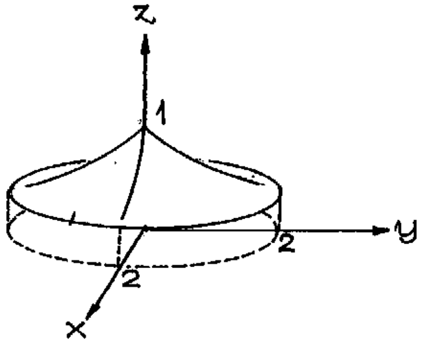
Funktionen $x\mapsto e^{-x}$ är bekant från envariabelkursen.



forts.

b) $r=\sqrt{x^2+y^2}$ är avståndet från punkten (x,y,z) till z-axeln.

c) $z=e^{-\sqrt{x^2+y^2}}$ är en hattformad yta symmetrisk m.a.p. z-axeln, som i figuren.

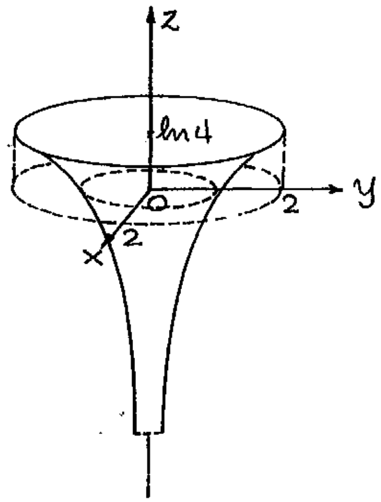


Anm. Ytan ovan påminner mycket om ett karuselltält.

Övning 1.14 (s. 2)

$z=\ln(x^2+y^2),\ 0<x^2+y^2\leq 4.$

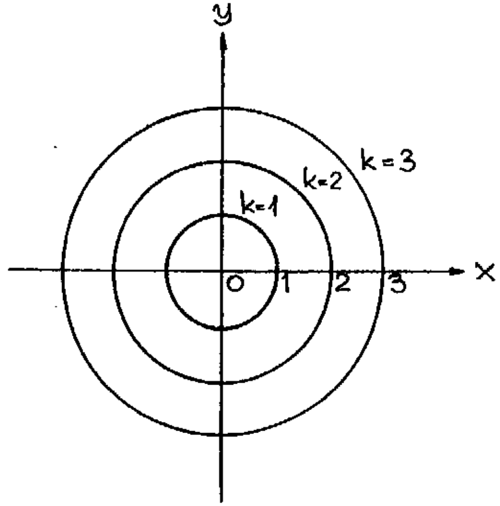
Vi sätter $r=\sqrt{x^2+y^2}$ och får den endimensionella funktionen $z=2\ln r,\ 0<r\leq 2.$ ln-funktionen är bekant från envariabelkursen. Den sökta ytan är en trattformad (alt. trumpetliknande) yta som i figuren.



Anm. Ytan har oändlig utsträckning nedåt.

Övning 1.15 (s. 2)

a) $f(x,y)=\sqrt{x^2+y^2}=k,\ k=1,2,3$ (Cirklar)

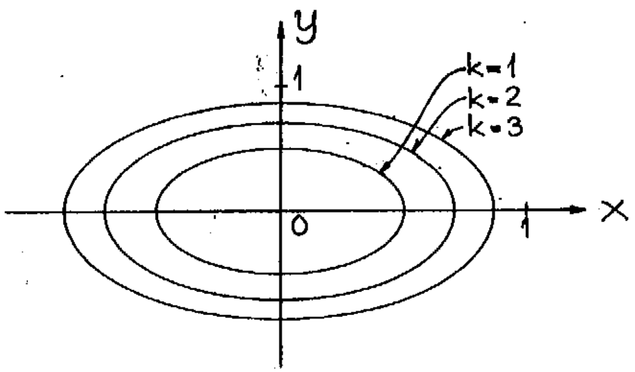


forts.

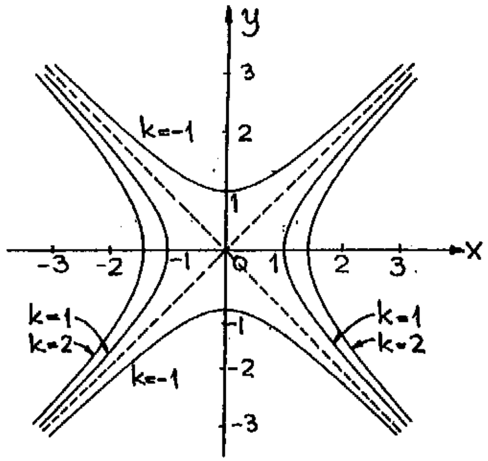
b) $f(x,y) = x^2+4y^2 = k$, $k=1,2,3.$

$$x^2+4y^2 = k \Leftrightarrow \frac{x^2}{k}+\frac{4y^2}{k}=1 \Leftrightarrow \frac{x^2}{(\sqrt{k})^2}+\frac{y^2}{(\sqrt{k}/2)^2}=1.$$

Nivåkurvorna är origocentriska ellipser med storaxeln dubbelt så stor som lillaxeln.



c) $f(x,y) = x^2-y^2 = k$, $k=-1,1,2$ (hyperbler).



Övning 1.16 (s.3)

$T(x,y) = 4x^2+2y^2 = K$, $K=0,4,8,12$

$$4x^2+2y^2=K \Leftrightarrow \frac{4x^2}{K}+\frac{2y^2}{K}=1 \Leftrightarrow \frac{x^2}{(\sqrt{K}/2)^2}+\frac{y^2}{(\sqrt{K}/\sqrt{2})^2}=1.$$

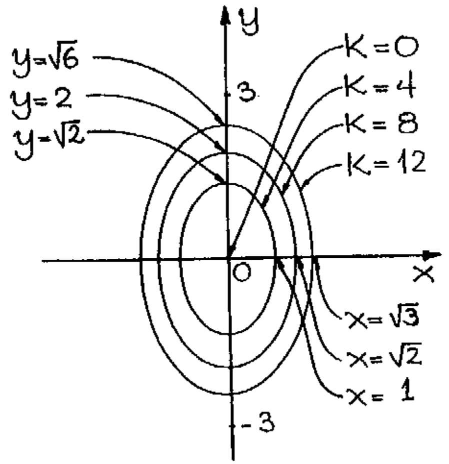
Isotermerna är tydligen ellipser.

$K=0 \Rightarrow 4x^2+2y^2=0 \Leftrightarrow (x,y)=(0,0)$ (degenererad).

$K=4 \Rightarrow \frac{x^2}{1^2}+\frac{y^2}{(\sqrt{2})^2}=1$;

$K=8 \Rightarrow \frac{x^2}{(\sqrt{2})^2}+\frac{y^2}{2^2}=1$;

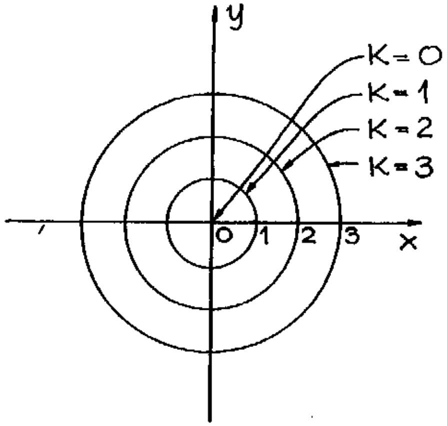
$K=12 \Rightarrow \frac{x^2}{(\sqrt{3})^2}+\frac{y^2}{(\sqrt{6})^2}=1$;



Övning 1.17 (s.3)

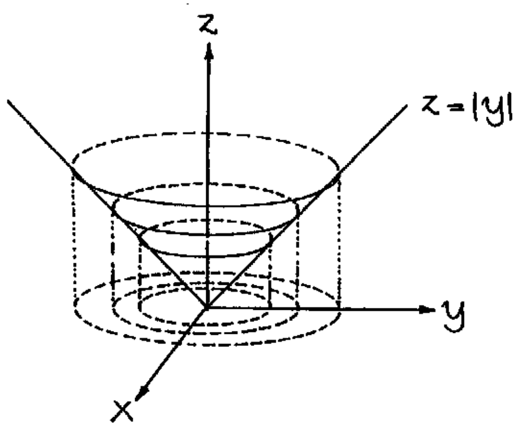
a) $f(x,y) = \sqrt{x^2+y^2} = K$;

Nivåkurvorna är origocentriska cirklar ($K>0$).



$\underline{x=0 \Rightarrow z=|y|}$; $\underline{y=0 \Rightarrow z=|x|}$.

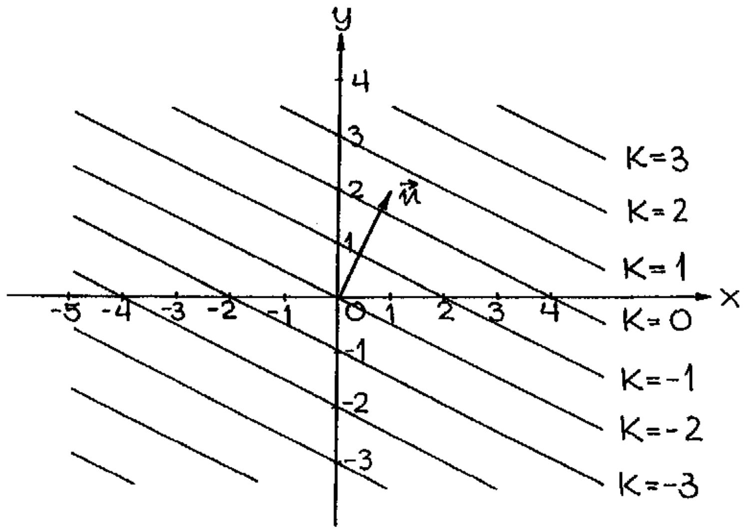
$z=\sqrt{x^2+y^2}$ är ekvationen för en rät cirkulär kon med spetsen i origo och symmetriaxel den positiva z-axeln. Spetsvinkeln är 90°



En K-kurva är skärningen mellan $z=\sqrt{x^2+y^2}$, den koniska ytan, och planet $z=K$. Det vi kallar nivåkurvor är deras projektion på xy-planet.

b) $\underline{f(x,y)=x+2y-2=K} \Leftrightarrow x+2y=K+2$;

Nivåkurvorna är linjer vinkelräta mot $\vec{n}=\begin{bmatrix}1\\2\end{bmatrix}$.



$z=x+2y-2 \Leftrightarrow \underline{x+2y-z=2}$.

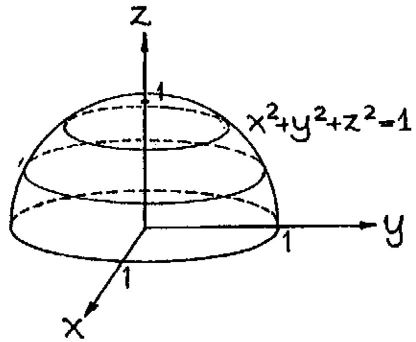
Ytan är planet med normalvektorn $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ gm punkten (0,1,0).

c) $\underline{f(x,y)=\sqrt{1-x^2-y^2}=K} \Leftrightarrow x^2+y^2=1-K^2$, $0\le K\le 1$.

Nivåkurvorna är tydligen origocentriska cirklar; samtliga har radie ≤ 1.

forts.

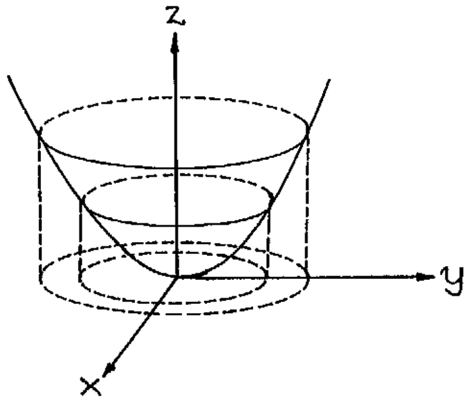
$z=\sqrt{1-x^2-y^2} \Leftrightarrow z^2=1-x^2-y^2 \wedge z\geq 0 \Leftrightarrow \underline{x^2+y^2+z^2=1 \wedge z\geq 0}$.



d) $\underline{f(x,y)=x^2+y^2=K}$, $K\geq 0$.

Nivåytorna är origocentriska cirklar.

$z=x^2+y^2$ är en rotationsparaboloid med toppen i origo och symmetriaxel den positiva z-axeln.

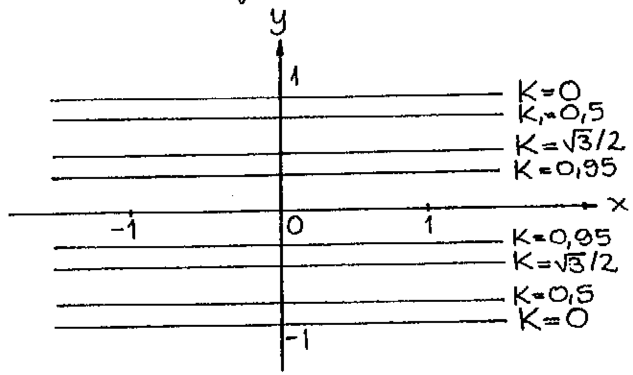


2-dimensionella ytor är svåra att skissera.

e) $f(x,y)=(1-y^2)^{1/2}=K \Leftrightarrow 1-y^2=K^2 \wedge 0\leq K\leq 1 \Leftrightarrow$

$\Leftrightarrow y^2=1-K^2 \wedge 0\leq K\leq 1 \Leftrightarrow \Leftrightarrow y=\pm\sqrt{1-K^2}$, $0\leq K\leq 1$

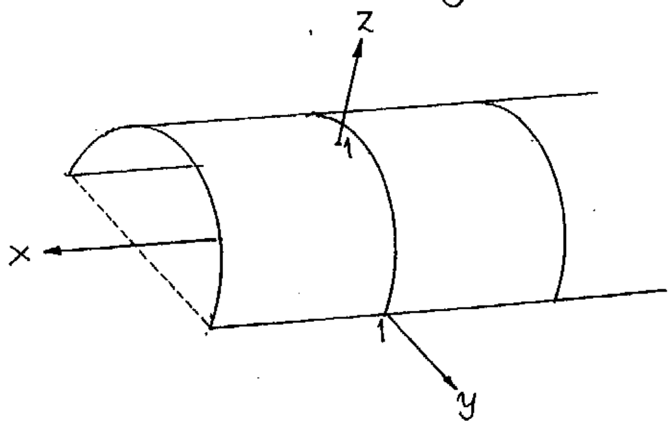
Nivåytorna är linjer parallella med x-axeln.



Se även Ex. 9 s.18 i grundboken.

$z=\sqrt{1-y^2} \Leftrightarrow z^2=1-y^2 \wedge z\geq 0 \Leftrightarrow y^2+z^2=1 \wedge z\geq 0$.

Ytan är en halv cylinder som i figuren nedan. (Se även Ex.7 i grundboken).



forts.

Anm. Cylinderytan är obegränsat lång. Detta är inte fallet i Ex. 7.

f) $f(x,y)=x=K,\ K\in\mathbb{R}.$

Nivåkurvorna är räta linjer parallella med y-axeln.

$z=x \Leftrightarrow x-z=0$ är ett plan vinkelrät mot $\vec{n}=(1,0,-1)$ genom punkten $(1,0,1)$.

Övning 1.18 (s. 3)

$T(x,y,z)=x^2+y^2+z^2+2x-2y \equiv (x+1)^2+(y-1)^2+z^2-2;$

$T(x,y,z)=K \Rightarrow (x+1)^2+(y-1)^2+z^2=K+2,\ (K>-2).$

Nivåytorna $T=0,1,2,3$ är koncentriska sfärer med centrum i $(-1,1,0)$ och radier $\sqrt{2},\sqrt{3},2$ resp. $\sqrt{5}$.

Övning 1.19 (s. 3)

$f(x,y,z)=x^2+y^2+2y+z^2-2z+3 \equiv x^2+(y+1)^2+(z-1)^2+1=K.$

$K=0 \wedge VL>0 \Rightarrow$ ingen nivåyta.

$K=1 \Rightarrow (x,y,z)=(0,-1,1)$; nivåytan = en punkt.

$K=2 \Rightarrow x^2+(y+1)^2+(z-1)^2=1$; sfär med centrum i punkten $(0,-1,1)$ och radien 1.

Övning 1.20 (s. 3)

$f(x,y,z)=z-\sqrt{1-x^2-y^2}=k,\ k=0,1.$

$k=0 \Rightarrow z=\sqrt{1-x^2-y^2} \Leftrightarrow x^2+y^2+z^2=1 \wedge z\geq 0.$

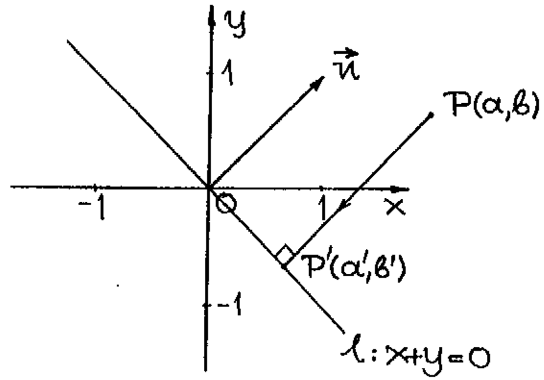
0-ytan är den övre halvan av enhetssfären.

$k=1 \Rightarrow z=1+\sqrt{1-x^2-y^2} \Leftrightarrow x^2+y^2+(z-1)^2=1 \wedge z\geq 1.$

1-ytan är den övre halvan av sfären $x^2+y^2+(z-1)^2=1.$

Övning 1.21 (s.3)

a)



$\vec{n}=\begin{pmatrix}1\\1\end{pmatrix}$ är en normalvektor till $l: x+y=0.$

Låt P vara en godtycklig punkt (utanför) l

och P' dess ortogonala projektion på l.

Normalen mot l genom P (och således gm P') ges i vektorform (obs! vektorer som kolonner)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} a+t \\ b+t \end{bmatrix}, \quad t \in \mathbb{R}$$

och i parameterform av

$$\begin{cases} x = a+t \\ y = b+t \end{cases}, \quad t \in \mathbb{R}.$$

Låt $t = t_0$ vara parametervärdet för P!

$x+y = a+t_0+b+t_0 = a+b+2t_0 = 0 \Leftrightarrow t_0 = -\frac{1}{2}(a+b) \Rightarrow$

$$\begin{cases} a' = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) \\ b' = b - \frac{1}{2}(a+b) = \frac{1}{2}(-a+b) \end{cases} \Leftrightarrow \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix};$$

Avbildningens matris är $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Låt nu (x, y) vara den löpande punkten och (u, v) bilden på l.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x-y \\ -x+y \end{bmatrix} \Leftrightarrow \begin{cases} u = \frac{1}{2}x - \frac{1}{2}y \\ v = -\frac{1}{2}x + \frac{1}{2}y \end{cases}.$$

b) Spegling på $l: x+y=0$ svarar mot parametervärdet $2t_0$ i föregående deluppgift.

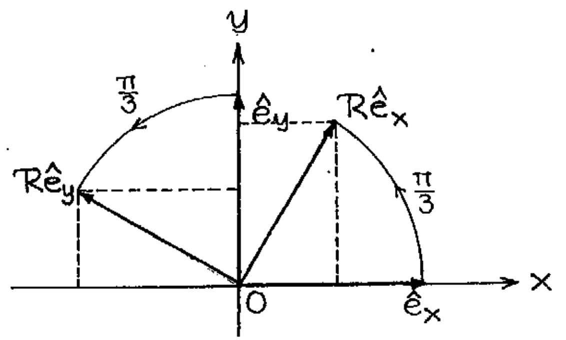
Låt $P''(a'', b'')$ vara spegelpunkten.

$$\begin{cases} a'' = a + 2t_0 = a - (a+b) = -b \\ b'' = b + 2t_0 = b - (a+b) = -a \end{cases} \Leftrightarrow \begin{bmatrix} a'' \\ b'' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix};$$

Avbildningens matris är $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ så att

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{cases} u = -y \\ v = -x \end{cases}.$$

c) En rotation är en linjär avbildning, så det räcker med att bestämma enhetsvektorernas bilder. Vi arbetar i standardbasen $(\hat{e}_x, \hat{e}_y)$.



Obs! Detta är linjär algebra!

$$\begin{cases} R\hat{e}_x = \cos\frac{\pi}{3}\hat{e}_x + \sin\frac{\pi}{3}\hat{e}_y = \frac{1}{2}\hat{e}_x + \frac{\sqrt{3}}{2}\hat{e}_y; \\ R\hat{e}_y = \cos(\frac{\pi}{3}+\frac{\pi}{2})\hat{e}_x + \sin(\frac{\pi}{3}+\frac{\pi}{2})\hat{e}_y = -\frac{\sqrt{3}}{2}\hat{e}_x + \frac{1}{2}\hat{e}_y; \end{cases}$$

Avbildningens matris har bildvektorerna

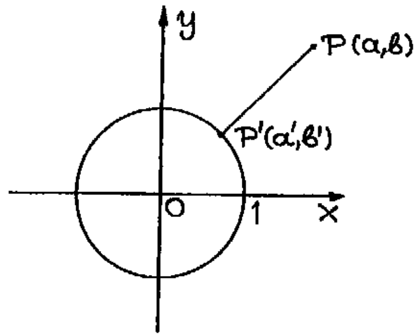
som kolonner.

$$[R]_e = (R\hat{e}_x\ R\hat{e}_y) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

Avbildningen ges alltså av

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} u = \frac{1}{2}x - \frac{\sqrt{3}}{2}y \\ v = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \end{cases}.$$

d)



$$P \mapsto P' \Rightarrow \begin{pmatrix} a' \\ b' \end{pmatrix} = k\begin{pmatrix} a \\ b \end{pmatrix} \Leftrightarrow \begin{cases} a' = ka \\ b' = kb \end{cases};$$

$$a'^2 + b'^2 = 1 \Rightarrow (ka)^2 + (kb)^2 = 1 \Leftrightarrow |k| = \frac{1}{\sqrt{a^2+b^2}};$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow \begin{cases} u = \frac{x}{\sqrt{x^2+y^2}} \\ v = \frac{y}{\sqrt{x^2+y^2}} \end{cases}, \quad (x,y) \neq (0,0).$$

Anm. Avbildningen ovan är centralprojektion.

Övning 1.22 (s. 3)

Ortogonalprojektion på en linje, spegling på en linje och rotation kring en punkt är alla linjära avbildningar; de representeras alla av konstanta matriser. Detta visas i den linjära algebran. Centralprojektion är däremot ingen linjär avbildning.

Anm. En avbildning $F: \mathbb{R}^n \to \mathbb{R}^p$ kallas linjär om $\forall u, v \in \mathbb{R}^n, \forall \lambda, \mu \in \mathbb{R}: F(\lambda u + \mu v) = \lambda F u + \mu F v$

I centralprojektion på enhetscirkeln avbildas varje multipel av en vektor på samma vektor.

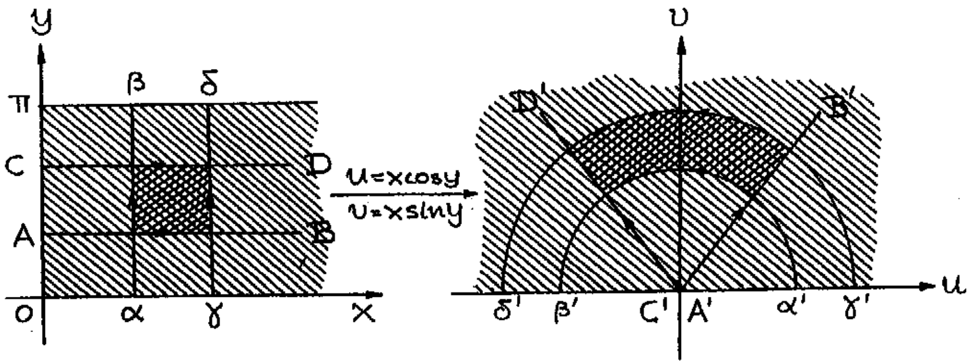
Övning 1.23 (s. 3)

$$\begin{cases} u = x\cos y \\ v = x\sin y \end{cases} \Leftrightarrow \begin{cases} u^2+v^2 = x^2 \\ \tan y = \frac{v}{u} \end{cases} \Leftrightarrow \begin{cases} \sqrt{u^2+v^2} = x \\ y = \arctan\frac{v}{u} \end{cases};$$

$x = k \Rightarrow \sqrt{u^2+v^2} = k$ (cirkelbågar).

$y = l \Rightarrow \frac{v}{u} = \tan l = m \Leftrightarrow v = m \cdot u$ (strålar).

På nästföljande sida illustreras avbildningen.



Övning 1.24 (s. 3)

$$\begin{cases} 5x^2+5xy+3y^2-8x-6y-3=0 \\ x=t^2 \\ y=t+1 \end{cases} \Rightarrow 5t^4+5t^2(t+1)+3(t+1)^2-$$

$$-8t^2-6(t+1)-3 = 5t^4+5t^3+5t^2+3t^2+6t+3-8t^2-6t-6+3=$$

$$= 5t^4+5t^3 = 5t^3(t+1)=0 \Leftrightarrow t=0 \vee t=-1\,;$$

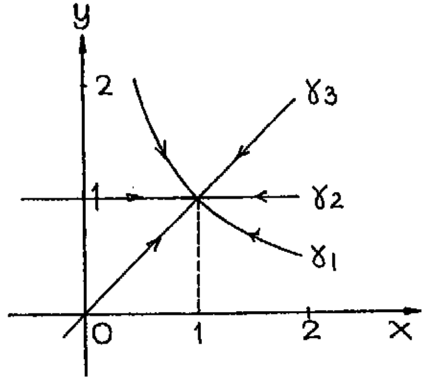
$$\begin{cases} x(0)=0 \wedge y(0)=1 \Rightarrow \underline{(x,y)=(0,1)}. \\ x(-1)=1 \wedge y(-1)=0 \Rightarrow \underline{(x,y)=(1,0)}. \end{cases}$$

Resultat: Kurvorna skär varandra i $(0,1)$, $(1,0)$.

Gransvärden och kontinuitet

Övning 1.25 (s.4)

a) Vi närmar oss punkten $(1,1)$ längs tre olika vägar γ_1, γ_2 och γ_3 som på figuren.



$\gamma_1: y=\frac{1}{x}$, $\gamma_2: y=1$, $\gamma_3: y=x$.

$$\forall (x,y)\in\gamma_1: \lim_{(x,y)\to(1,1)} \frac{xy-1}{x-1}\begin{bmatrix} x=t \\ y=1/t\end{bmatrix} = \lim_{t\to 1} 0 = 0.$$

$$\forall (x,y)\in\gamma_2: \lim_{(x,y)\to(1,1)} \frac{xy-1}{x-1}\begin{bmatrix} x=t \\ y=1\end{bmatrix} = \lim_{t\to 1} \frac{t-1}{t-1} = 1.$$

$$\forall (x,y)\in\gamma_3: \lim_{(x,y)\to(1,1)} \frac{xy-1}{x-1}\begin{bmatrix} x=t \\ y=t\end{bmatrix} = \lim_{t\to 1} \frac{t^2-1}{t-1} = 2.$$

Tre olika vägar $\Rightarrow$ tre olika resultat.

Gränsvärdet existerar inte.

b) Från envaribelanalysen vet vi att

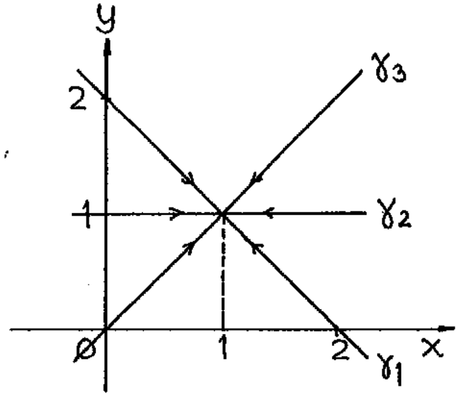
$$\lim_{x\to 0^+} x\ln x = 0.$$

Mer allmänt gäller $\lim_{x\to 0} x^\alpha \ln x = 0$, $\alpha>0$.

$$\mathbf{x}=(x,y) \Rightarrow |\mathbf{x}| = \sqrt{x^2+y^2} \Rightarrow \lim_{|\mathbf{x}|\to 0} |\mathbf{x}|^2 \ln|\mathbf{x}|^2 =$$

$$= \lim_{|\mathbf{x}|\to 0} 2|\mathbf{x}|^2 \ln|\mathbf{x}| = 0 = \lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2).$$

c) Vi närmar oss $(1,1)$ längs 3 olika vägar, som i figuren nedan.



$\gamma_1: y = 2-x$, $\gamma_2: y=1$, $\gamma_3: y=x$.

$$\forall (x,y)\in\gamma_1: \lim_{(x,y)\to(1,1)} \frac{x-y}{x-1}\begin{bmatrix} x=t \\ y=2-t\end{bmatrix} = \lim_{t\to} \frac{2\cancel{(t-1)}}{\cancel{t-1}} = 2.$$

$$\forall (x,y)\in\gamma_2: \lim_{(x,y)\to(1,1)} \frac{x-y}{x-1}\begin{bmatrix} x=t \\ y=1\end{bmatrix} = \lim_{t\to 1} \frac{\cancel{t-1}}{\cancel{t-1}} = 1.$$

$$\forall (x,y)\in\gamma_3: \lim_{(x,y)\to(1,1)} \frac{x-y}{x-1}\begin{bmatrix} x=t \\ y=t\end{bmatrix} = \lim_{t\to 1} 0 = 0.$$

Tre olika vägar ger tre olika resultat; gränsvärdet existerar således inte.

d) Vi närmar oss origo längs räta linjer $y=kx$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{2x^2+y^2}\left[y=kx\right] = \lim_{x\to 0} \frac{x^2+2k^2x^2}{2x^2+k^2x^2} = \lim_{x\to 0}\frac{1+2k^2}{2+k^2}$$

"Gränsvärdet" beror av k! Något egentligt gränsvärde existerar således inte.

e) <u>Anm.</u> Om funktionen f är definierad på ett intervall $[a,b]$ (med a och b ändliga), så har f ett största och ett minsta funktionsvärde på detta intervall. Detta bör vara bekant från envariabelanalysen.

$$f(x,y) = \frac{x^3-x^2y}{x^2+y^2+xy} \Rightarrow f(r\cos\theta, r\sin\theta) = r\cdot\frac{\cos^2\theta(1-\sin\theta)}{1+\frac{1}{2}\sin 2\theta};$$

Funktionen $\phi(\theta) = \dfrac{\cos^2\theta(1-\sin\theta)}{1+\frac{1}{2}\sin 2\theta}$, $0\le\theta\le 2\pi$, är uppenbarligen kontinuerlig (nämnaren blir aldrig 0), så den antar både minsta och största värdet m resp. M. Vi har alltså

$$m\cdot r \le f(r\cos\theta, r\sin\theta) \le Mr \Leftrightarrow m|\mathbf{x}| \le f(x,y) \le M|\mathbf{x}|. \quad (*)$$

$$\lim_{|\mathbf{x}|\to 0} m|\mathbf{x}| = 0 = \lim_{|\mathbf{x}|\to 0} M|\mathbf{x}| \overset{(*)}{\Rightarrow} \lim_{(x,y)\to(0,0)} f(x,y) = 0,$$

enligt instängningsprincipen. Med $|\mathbf{x}|$ menas förstås $r = |\mathbf{x}| = \sqrt{x^2+y^2}$ = avståndet från origo.

f) $f(x,y)=(1+x^2+y^2)\exp\left\{\frac{1}{x^2+y^2+x^2y}\right\}$

$$f(r\cos\theta, r\sin\theta)=(1+r^2)\exp\left\{\frac{1}{r^2(1+r\cos^2\theta\sin\theta)}\right\}=$$
$$=\exp\left\{\frac{\ln(1+r^2)}{r^2}\cdot\frac{1}{1+r\cos^2\theta\sin\theta}\right\}$$

Vi väljer en liten omgivning av origo, $|\mathbf{x}|<\delta\ll 1$.

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{|\mathbf{x}|\to 0} f(\mathbf{x})=\lim_{r\to 0} f(r\cos\theta, r\sin\theta)=$$
$$=\lim_{r\to 0}\exp\left\{\frac{\ln(1+r^2)}{r^2}\cdot\frac{1}{1+r\cos^2\theta\sin\theta}\right\}=(u=e^{\ln u}, u>0)=$$
$$=\exp\left\{\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2}\cdot\frac{1}{1+r\cos^2\theta\sin\theta}\right\}=$$
$$=\exp\left\{\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2}\cdot\lim_{r\to 0}\frac{1}{1+r\cos^2\theta\sin\theta}\right\}=$$
$$=\exp\left\{\lim_{r\to 0}\frac{r^2+O(r^4)}{r^2}\cdot\lim_{r\to 0}\frac{1}{1+r\cos^2\theta\sin\theta}\right\}=$$
$$=\exp\left\{\lim_{r\to 0}(1+O(r^2))\cdot\lim_{r\to 0}\frac{1}{1+r\cos^2\theta\sin\theta}\right\}=$$
$$=\exp\{1\cdot 1\}=\underline{e}.$$

g) $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}}\frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}=\lim_{|\mathbf{x}|\to 0}\frac{\sin|\mathbf{x}|}{|\mathbf{x}|}=1$, (standard-gränsvärde).

Övning 1.26 (s. 4)

a) Vi betraktar en liten omgivning av origo

$$O_\delta=\{\mathbf{x}\in\mathbb{R}^3:\ |\mathbf{x}|<\delta<1\}.$$

$$\lim_{\mathbf{x}\to\mathbf{0}}\frac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2+x_1x_2x_3}\left[x_1=r\sin\theta\cos\varphi,\ x_2=r\sin\theta\sin\varphi,\ x_3=r\cos\theta\right]$$
$$=\lim_{r\to 0}\frac{\sin r^2}{r^2+r^3\sin^2\theta\cos\theta\sin\varphi\cos\varphi}=$$
$$=\lim_{r\to 0}\frac{\sin r^2}{r^2}\cdot\lim_{r\to 0}\frac{1}{1+r\sin^2\theta\cos\theta\sin\varphi\cos\varphi}=$$
$$=\lim_{r\to 0}\frac{\sin r^2}{r^2}[u=r^2]=\lim_{u\to 0^+}\frac{\sin u}{u}=1.$$

b) Vi arbetar i omgivningen $O_\delta=\{\mathbf{x}\in\mathbb{R}^3: |\mathbf{x}|<\delta<\frac{1}{3}\}$.

$$\lim_{\mathbf{x}\to\mathbf{0}}\frac{e^{|\mathbf{x}|^2}-1}{|\mathbf{x}|^2+x_1^2x_2+x_2^2x_3+x_3x_1}\ (\text{sfäriska koordinater})=$$
$$=\lim_{r\to 0}\frac{e^{r^2}-1}{r^2(1+r(\sin^3\theta\cos^2\varphi\sin\varphi+\sin^2\theta\sin^2\varphi\cos\varphi+\sin\theta\cos^2\theta\cos\varphi))}$$
$$=\lim_{r\to 0}\frac{e^{r^2}-1}{r^2}\cdot\lim_{r\to 0}\frac{1}{1+r\cdot f(\theta,\varphi)}=\lim_{r\to 0}\frac{e^{r^2}-1}{r^2}=1$$

$\underline{\text{Anm.}}$ $\displaystyle\lim_{r\to 0}\frac{e^{r^2}-1}{r^2}=\lim_{r\to 0}\frac{r^2+O(r^4)}{r^2}=\lim_{r\to 0}(1+O(r^2))=1.$

c) $\displaystyle\lim_{\mathbf{x}\to\mathbf{0}}\frac{\ln(1+|\mathbf{x}|^2)}{|\mathbf{x}|^2+\sin(x_1x_2x_3)}$ (sfäriska koordinater) $=$

$$=\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2+\sin(r^3(\sin^3\theta\cos\theta\sin\varphi\cos\varphi))}=$$
$$=\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2}\cdot\frac{1}{1+\frac{\sin(r^3(\sin^3\theta\cos\theta\sin\varphi\cos\varphi))}{r^2}}=$$
$$=\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2}\cdot\lim_{r\to 0}\frac{1}{1+r\cdot\frac{\sin(r^3(\sin^3\theta\cos\theta\sin\varphi\cos\varphi))}{r^3}}=$$
$$=\lim_{r\to 0}\frac{\ln(1+r^2)}{r^2}=\lim_{r\to 0}\frac{r^2+O(r^4)}{r^2}=\lim_{r\to 0}(1+O(r^2))=1.$$

$\underline{\text{Anm.}}$ $\displaystyle\lim_{r\to 0}\frac{\sin(r^3(\sin^3\theta\cos\theta\sin\varphi\cos\varphi))}{r^3}$ begränsad.

Övning 1.27 (s.4)

a) Vi sätter $\mathbf{x}=(x,y)$, $\mathbf{a}=(a,b)$.

$$|\mathbf{x}-\mathbf{a}| = |(x,y)-(a,b)| = |(x-a,y-b)| = \sqrt{(x-a)^2+(y-b)^2};$$

$f(\mathbf{x})\to A$ då $\mathbf{x}\to\mathbf{a} \Leftrightarrow \forall\varepsilon>0, \exists\delta>0: 0<|\mathbf{x}-\mathbf{a}|<\delta \Rightarrow$

$\Rightarrow |f(\mathbf{x})-A|<\varepsilon.$

b) $$0 \le \left|\frac{x^4y}{x^2+(x+y)^2}\right| \le \frac{x^4|y|}{x^2} = x^2|y| \xrightarrow[|\mathbf{x}|\to 0]{} 0;$$

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^4y}{x^2+(x+y)^2} = 0.$$

Övning 1.28 (s.4)

a) $$f(x,y) = \frac{\ln(x^2+y^2)}{x^2+y^2+xy} \Rightarrow f(r\cos\theta, r\sin\theta) = \frac{\ln r^2}{r^2(1+\frac{1}{2}\sin 2\theta)};$$

$$-1\le \sin 2\theta \le 1 \Leftrightarrow -\tfrac{1}{2} \le \tfrac{1}{2}\sin 2\theta \le \tfrac{1}{2} \Leftrightarrow \tfrac{1}{2} \le 1+\tfrac{1}{2}\sin 2\theta \le \tfrac{3}{2} \Leftrightarrow$$

$$\Leftrightarrow \frac{2}{3} \le \frac{1}{1+\frac{1}{2}\sin 2\theta} \le 2 \Rightarrow (\theta\text{-delen begränsad}) \Rightarrow$$

$$\Rightarrow \frac{2}{3}\frac{\ln|\mathbf{x}|^2}{|\mathbf{x}|^2} \le f(x,y) \le 2\frac{\ln|\mathbf{x}|^2}{|\mathbf{x}|^2}; \quad (*)$$

$$\lim_{|\mathbf{x}|\to\infty}\frac{\ln|\mathbf{x}|^2}{|\mathbf{x}|^2} = 0 \overset{(*)}{\Rightarrow} \lim_{|\mathbf{x}|\to\infty}\frac{\ln(x^2+y^2)}{x^2+y^2+xy} = 0$$ (enligt instängningsprincipen).

b) $$\lim_{|\mathbf{x}|\to\infty}\frac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2} = 0,$$ ty $$\left|\frac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2}\right| \le \frac{1}{|\mathbf{x}|^2} \xrightarrow[|\mathbf{x}|\to\infty]{} 0$$ och instängningsprincipen från envariabelanalysen.

c) $$|x|\le|\mathbf{x}| \wedge |y|\le|\mathbf{x}| \Rightarrow \left|\frac{x^2y}{(x^2+y^2)^2+x^2y^2}\right| =$$

$$= \frac{x^2|y|}{(x^2+y^2)^2+x^2y^2} \le \frac{x^2|y|}{(x^2+y^2)^2} \le \frac{|\mathbf{x}|^2\cdot|\mathbf{x}|}{|\mathbf{x}|^4} = \frac{1}{|\mathbf{x}|} \xrightarrow[|\mathbf{x}|\to\infty]{} 0 \Rightarrow$$

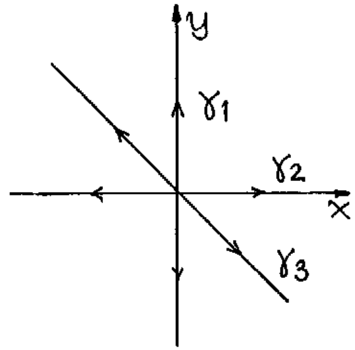
$$\Rightarrow \lim_{|\mathbf{x}|\to\infty}\frac{x^2y}{(x^2+y^2)^2+x^2y^2} = 0.$$

d) $$|x|\le|\mathbf{x}|^2 \wedge |y|\le|\mathbf{x}|^2 \Rightarrow |xye^{-(x^2+y^2)}| \le |xy|e^{-|\mathbf{x}|^2}$$

$$= |x|\cdot|y|e^{-|\mathbf{x}|^2} \le |\mathbf{x}|^2\cdot e^{-|\mathbf{x}|^2} \xrightarrow[x\to\infty]{} 0 \Rightarrow \lim_{|\mathbf{x}|\to\infty} xye^{-|\mathbf{x}|^2} = 0.$$

Anm. $\lim_{u\to\infty}\frac{u}{e^u} = 0$, från envariabelanalysen.

e) Vi prövar olika vägar och ser vad som händer.



γ_1: y-axeln, γ: x-axeln, γ_3: $y=-x$.

$$\forall(x,y)\in\gamma_1: \lim_{|\mathbf{x}|\to\infty} xye^{-(x+y)^2} = \lim_{|\mathbf{x}|\to\infty} 0\cdot ye^{-y^2} = 0.$$

$$\forall(x,y)\in\gamma_2: \lim_{|\mathbf{x}|\to\infty} xye^{-(x+y)^2} = \lim_{|\mathbf{x}|\to\infty} x\cdot 0e^{-x^2} = 0.$$

$$\forall(x,y)\in\gamma_3: \lim_{|\mathbf{x}|\to\infty} xye^{-(x+y)^2} = \lim_{x\to\infty}(-x^2) = -\infty.$$

Resultat: Gränsvärdet existerar inte.

Övning 1.30 (s. 5)

a) $f(x,y)$ är kontinuerlig.

$$\lim_{|x|\to 0} \frac{\sin|x|^2}{|x|^2} \left[u=|x|^2\right] = \lim_{u\to 0} \frac{\sin u}{u} = 1.$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y)\neq(0,0) \\ 1, & (x,y)=(0,0) \end{cases}$$ kontinuerlig utvidgning av $f(x,y)$.

b) $f(x,y)$ är kontinuerlig utanför axlarna.

$$\lim_{|x|\to 0} \frac{\sin xy}{xy+x^3y^3} = \lim_{|x|\to 0} \frac{\sin xy}{xy} \cdot \lim_{|x|\to 0} \frac{1}{1+x^2y^2} = 1\cdot 1 = 1;$$

$$F(x,y) = \begin{cases} f(x,y), & xy\neq 0 \\ 1, & xy=0 \end{cases}$$ kontinuerlig utvidgning av $f(x,y)$.

Anm. $xy=0 \Leftrightarrow x=0 \vee y=0$ (koordinataxlarna).

c) $f(x,y)$ är kontinuerlig (nämnaren $\neq 0$).

$$\lim_{|x|\to 0} \frac{(x+y)^4}{x^2+y^2} [y=kx] = \lim_{x\to 0} \frac{(1+k)^2\cancel{x^2}}{(1+k^2)\cancel{x^2}} = \lim_{x\to 0} \frac{(1+k)^2}{1+k^2}$$

beror av k; gränsvärdet existera alltså inte.

Det går således inte att utvidga f till en kontinuerlig funktion i hela planet.

d) f är kontinuerlig i sin definitionsmängd.

$$\lim_{|x|\to 0} \frac{(x+y)^4}{x^2+y^2} \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \lim_{r\to 0} r^2\cdot(\sin\theta+\cos\theta)^4 = 0.$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$ kontinuerlig utvidgning av $f(x,y)$.

e) f är kontinuerlig i sin definitionsmängd.

$$\lim_{|x|\to 0} x\exp\left\{-\frac{1}{\sqrt{x^2+y^2}}\right\} [y=kx] = \lim_{x\to 0} x\exp\left\{\frac{-1}{\sqrt{1+k^2}}\cdot\frac{1}{x}\right\} =$$

$$= \lim_{u\to\infty} \frac{1}{u} e^{-cu} = 0, \quad c = 1/\sqrt{1+k^2}.$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$ kontinuerlig utvidgning av $f(x,y)$.

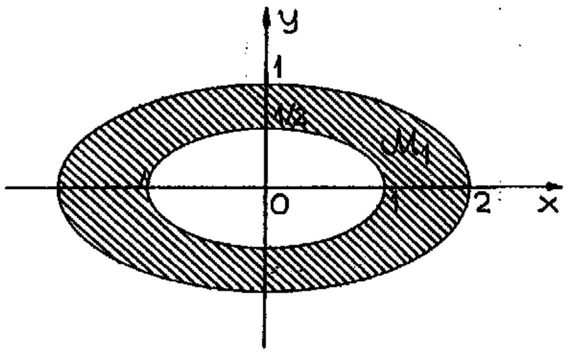
Anm. Man kan även pröva med polär substitution vid gränsvärdesbestämningen.

Resultat: I samtliga fall utom i c) går det att bestämma en kontinuerlig utvidgning.

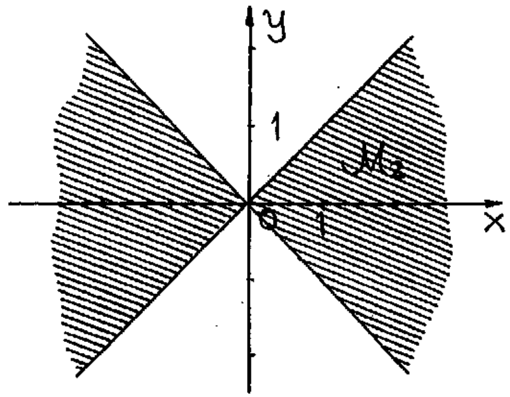
Blandade problem

Övning 1.31 (s. 5)

$$1\le x^2+4y^2\le 4 \Leftrightarrow \begin{cases}1\le x^2+4y^2\\ x^2+4y^2\le 4\end{cases} \Leftrightarrow \begin{cases}1\le \dfrac{x^2}{1^2}+\dfrac{y^2}{(1/2)^2}\\ \dfrac{x^2}{2^2}+\dfrac{y^2}{1^2}\le 1\end{cases};$$

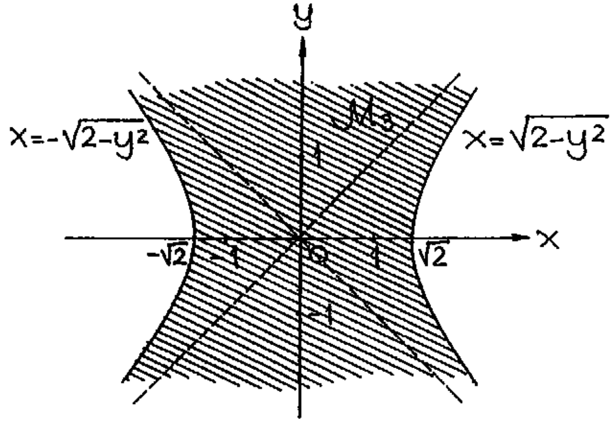


$$x^2-y^2\ge 0 \Leftrightarrow y^2\le x^2 \Leftrightarrow |y|\le|x| \Leftrightarrow -|x|\le y\le |x|.$$

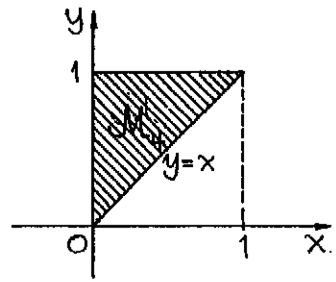


Anm. M_2 är hela det skuggade området.

$$x^2-y^2\le 2 \Leftrightarrow \frac{x^2}{(\sqrt2)^2}-\frac{y^2}{(\sqrt2)^2}\le 1 \quad \text{(Se fig. nästa sida.)}$$

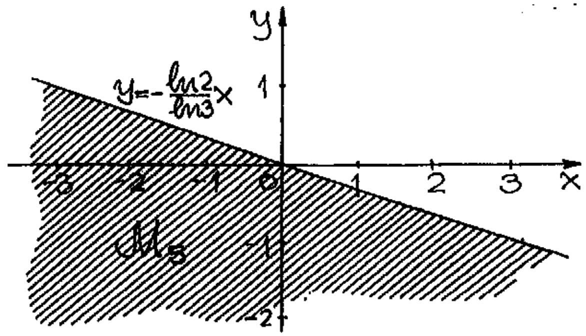


$$0\le x\le y\le 1 \Leftrightarrow \underline{x\le y} \wedge \underline{0\le x\le 1} \wedge \underline{0\le y\le 1}.$$



$$2^x\cdot 3^y\le 1 \Leftrightarrow e^{x\ln 2}\cdot e^{y\ln 3}\le 1 \Leftrightarrow e^{x\ln2+y\ln3}\le 1$$

$$\Leftrightarrow x\ln2+y\ln3\le 0 \Leftrightarrow y\ln3\le -x\ln2 \Leftrightarrow \underline{y\le -\frac{\ln2}{\ln3}x}.$$



Övning 1.32 (s.5)

a) $\lim\limits_{(x_1,x_2)\to 0} \dfrac{x^2 \ln x_1}{(x_1-1)^2+x_2^2} \begin{bmatrix} u = x_1-1 \\ v = x_2 \end{bmatrix} = \lim\limits_{(u,v)\to 0} \dfrac{u^2 \ln u}{u^2+v^2} \begin{bmatrix} u = r\cos\theta \\ v = r\sin\theta \end{bmatrix}$

$= \lim\limits_{r\to 0} \dfrac{r^2\cos^2\theta \ln(1+r\cos\theta)}{r^2} = \lim\limits_{r\to 0} \cos^2\theta \ln(1+r\cos\theta) = 0.$

Anm. I den polära substitution ska $r<1$.

b) $\lim\limits_{(x_1,x_2)\to 0} \dfrac{x^2+y^2}{x^2+xy+y^2} \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix} = \lim\limits_{r\to 0} \dfrac{1}{1+\sin\theta\cos\theta};$

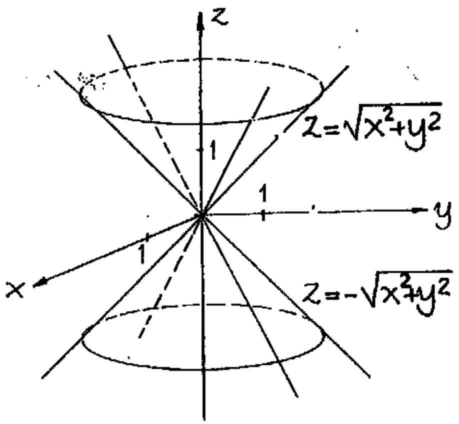
Gränsvärdet existerar inte.

Övning 1.33 (s.5)

a) $M_1 = \{(x,y,z):\ x^2+y^2-z^2 \leq 0\}.$

$x^2+y^2-z^2 \leq 0 \Leftrightarrow x^2+y^2 \leq z^2 \Leftrightarrow \sqrt{x^2+y^2} \leq |z| \Leftrightarrow$

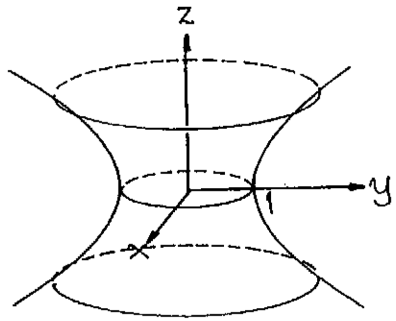
$\Leftrightarrow z \geq \sqrt{x^2+y^2} \ \vee\ z \leq -\sqrt{x^2+y^2};$



M_1 består av alla punkter på och innanför dubbelkonen $|z| = \sqrt{x^2+y^2}$.

b) $x^2+y^2-z^2 \leq 1 \Leftrightarrow x^2+y^2-1 \leq z^2 \Leftrightarrow \sqrt{x^2+y^2-1} \leq |z|$

M_2 består av punkterna på och "innanför" (rotations)hyperboloiden $|z| = \sqrt{x^2+y^2-1}$.



Övning 1.34 (s.6)

$f(x,y,z) = x^2+y^2-z^2-1 = k,\ k=-1,0.$

$k=-1 \Rightarrow x^2+y^2-z^2 = 0 \Leftrightarrow z^2 = x^2+y^2 \Leftrightarrow |z| = \sqrt{x^2+y^2}.$

Ytan är en tvåmantlad (rotations)kon.

Anm. Normalt säger man inte tvåmantlad kon; man säger dubbelkon i stället.

$k=0 \Rightarrow x^2+y^2-z=1 \Leftrightarrow z^2 = x^2+y^2-1 \Leftrightarrow |z| = \sqrt{x^2+y^2-1};$

en enmantlad (rotations)hyperboloid.

Övning 1.35 (s. 6)

$x^2-2x+y^2-4y+z^2+2z+3 \equiv (x-1)^2+(y-2)^2+(z+1)^2-6+3=0$

$\Leftrightarrow \underline{(x-1)^2+(y-2)^2+(z+1)^2=(\sqrt{3})^2}.$

Sfären har medelpunkten $(1,2,-1)$ och radien $\sqrt{3}$. Jag ska visa att avståndet från sfärens medelpunkt till planet är lika med radien $\sqrt{3}$.

$$x+y+z-5=0 \Leftrightarrow \frac{x+y+z-5}{\sqrt{3}}=0 \Rightarrow d=\frac{|1+2-1-5|}{\sqrt{3}}=\sqrt{3}.$$

Planet tangerar alltså sfären i (λ,μ,ν) säg. En normal till planet genom sfärens medelpunkt ges i vektorform av $(x,y,z)^T=(1,2,-1)^T+$ $+t\cdot(1,1,1)^T=(1+t,2+t,-1+t)^T$; $(1,1,1)^T$ är en normalvektor till planet. Låt $t=t_0$ svara mot (λ,μ,ν).

$$\begin{cases}\lambda=1+t_0\\ \mu=2+t_0\\ \nu=-1+t\end{cases} \Rightarrow \lambda+\mu+\nu-5=2+3t_0-5=3t_0-3=0 \Rightarrow t_0=1.$$

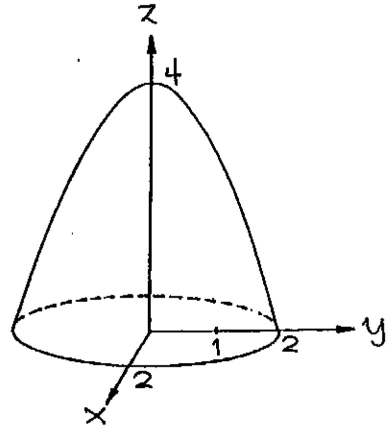
Tangeringspunkten har koordinaterna $(2,3,0)$.

Avståndsformeln genomgås i den analytiska geometrin men även i den linjära algebran. Den finns i BETA och på sidan 85.

Övning 1.36 (s. 6)

a) $z=f(x,y)=\underline{4-x^2-y^2=k} \Leftrightarrow x^2+y^2=4-k,\ k\le 4.$

Nivåytorna är origocentriska cirklar med radier $\le\sqrt{4-k}$; $x=0 \Rightarrow z=4-y^2$ (parabel) och $y=0 \Rightarrow z=4-x^2$ (parabel). Ytan är en stympad uppochnedvänd rotationsparaboloid.



b) $f(x,y)=\sqrt{4-x^2-y^2}=z \Leftrightarrow x^2+y^2+z^2=4 \wedge z\ge 0.$

Ytan är en sfär med medelpunkt $(0,0,0)$ och radien 2. Endast dess övre halva ingår i grafen, som finns i facit.

c) $z=x^2-y^2$, $|x|\le 2$, $|y|\le 2$, är ett stycke enmantlad hyperboloid; dess graf finns i facit.

2. Differentialkalkyl av reellvärda funktioner

Partiella derivator

Övning 2.1 (s.25)

a) $f(x,y)=x+x^3y+x^2y^3+y^5$

$$\begin{cases} \frac{\partial f}{\partial x}=1+3x^2\cdot y+2x\cdot y^3+0\cdot y^5=1+3x^2y+2xy^3. \\ \frac{\partial f}{\partial y}=0+x^3\cdot 1+x^2\cdot 3y^2+5y^4=x^3+3x^2y^2+5y^4. \end{cases}$$

b) $f(x,y)=(xy^2+1)^5$

$f(x,y)=u^5 \wedge u=xy^2+1;$

$$\begin{cases} \frac{\partial f}{\partial x}=5u^4\cdot\frac{\partial u}{\partial x}=5(xy^2+1)^4\cdot y^2=5y^2(x^2+1)^4. \\ \frac{\partial f}{\partial y}=5u^4\frac{\partial u}{\partial y}=5(xy^2+1)^4\cdot 2xy=10xy(xy^2+1)^4. \end{cases}$$

c) $f(x,y)=\frac{x+y}{x-y};$

$$D_x f(x,y)=D_x\frac{x+y}{x-y}=\frac{(x-y)D_x(x+y)-(x+y)D_x(x-y)}{(x-y)^2}=$$

$$=\frac{(x-y)\cdot 1-(x+y)\cdot 1}{(x-y)^2}=\frac{x-y-x-y}{(x-y)^2}=-\frac{2y}{(x-y)^2}.$$

$$D_y f(x,y)=D_y\frac{x+y}{x-y}=\frac{(x-y)D_y(x+y)-(x+y)D_y(x-y)}{(x-y)^2}=$$

$$=\frac{(x-y)\cdot 1-(x+y)(-1)}{(x-y)^2}=\frac{x-y+x+y}{(x-y)^2}=\frac{2x}{(x-y)^2}.$$

d) $f(x,y)=\arctan\frac{y}{x}$

$f(x,y)=\arctan u,\ u=\frac{y}{x};$

$$\frac{\partial f}{\partial x}=\frac{1}{1+u^2}\frac{\partial u}{\partial x}=\frac{1}{1+y^2/x^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}.$$

$$\frac{\partial f}{\partial y}=\frac{1}{1+u^2}\frac{\partial u}{\partial y}=\frac{1}{1+y^2/x^2}\cdot\frac{1}{x}=\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}.$$

e) $f(x,y)=\ln\sqrt{x^2+y^2}.$

$f(x,y)=\frac{1}{2}\ln u,\ u=x^2+y^2.$

$$\begin{cases} \frac{\partial f}{\partial x}=\frac{1}{2u}\frac{\partial u}{\partial x}=\frac{1}{2u}\cdot 2x=\frac{x}{u}=\frac{x}{x^2+y^2}. \\ \frac{\partial f}{\partial y}=\frac{1}{2u}\frac{\partial u}{\partial y}=\frac{1}{2u}\cdot 2y=\frac{y}{u}=\frac{y}{x^2+y^2}. \end{cases}$$

Övning 2.2 (s.25)

a) $f(x_1,x_2,x_3)=\ln|x_1x_2+x_2x_3+x_3x_1|.$

$f(x_1,x_2,x_3)=\ln|u|,\ u=x_1x_2+x_2x_3+x_3x_1;$

$$\begin{cases} \frac{\partial f}{\partial x_1}=\frac{1}{u}\frac{\partial u}{\partial x_1}=\frac{1}{u}(x_2+x_3)=\frac{x_2+x_3}{x_1x_2+x_2x_3+x_3x_1}. \\ \frac{\partial f}{\partial x_2}=\frac{1}{u}\frac{\partial u}{\partial x_2}=\frac{1}{u}(x_1+x_3)=\frac{x_1+x_3}{x_1x_2+x_2x_3+x_3x_1}. \\ \frac{\partial f}{\partial x_3}=\frac{1}{u}\frac{\partial u}{\partial x_3}=\frac{1}{u}(x_1+x_2)=\frac{x_1+x_2}{x_1x_2+x_2x_3+x_3x_1}. \end{cases}$$

b) $f(x_1,x_2,x_3)=\sqrt{x_1^2+x_2^2+x_3^2}.$

$f(x_1,x_2,x_3)=\sqrt{u},\ u=x_1^2+x_2^2+x_3^2;$

$$\begin{cases} \frac{\partial f}{\partial x_1}=\frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x_1}=\frac{1}{2\sqrt{u}}\cdot 2x_1=\frac{x_1}{\sqrt{u}}=\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}. \\ \frac{\partial f}{\partial x_2}=\frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x_2}=\frac{1}{2\sqrt{u}}\cdot 2x_2=\frac{x_2}{\sqrt{u}}=\frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}. \\ \frac{\partial f}{\partial x_3}=\frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x_3}=\frac{1}{2\sqrt{u}}\cdot 2x_3=\frac{x_3}{\sqrt{u}}=\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}. \end{cases}$$

c) $f(x_1,x_2,x_3)=x_1^{(x_2^{x_3})};$

$$f(x) = x_1^{(x_2^{x_3})} = e^{(x_2^{x_3})\ln x_1} = e^{u} \wedge u = (x_2^{x^3})\ln x_1.$$

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = e^{u}\dfrac{\partial u}{\partial x_1} = e^{u}\cdot\left(x_2^{x^3}\cdot\dfrac{1}{x}\right) = x_2^{x_3}\cdot x_1^{(x_2^{x_3})-1}. \\ \dfrac{\partial f}{\partial x_2} = e^{u}\dfrac{\partial u}{\partial x_2} = e^{u}(x_3\cdot x_2^{x_3-1})\ln x_1 = x_1^{(x_2^{x_3})}x_3\cdot x_2^{x_3-1}\ln x_1. \\ \dfrac{\partial f}{\partial x_3} = e^{u}\dfrac{\partial u}{\partial x_3} = e^{u}\cdot(x_2^{x_3})\ln x_2\cdot\ln x_1 = x_1^{(x_2^{x_3})}x_2^{x_3}\ln x_1\cdot\ln x_2 \end{cases}$$

Övning 2.3 (s. 25)

$f(x,y) = x^{y},\ x > 0.$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow y x^{y-1} = x^{y}\ln x \Leftrightarrow \frac{y}{x}\cdot x^{y} = x^{y}\cdot\ln x \Leftrightarrow$$

$$\Leftrightarrow \frac{y}{x} = \ln x \Leftrightarrow \underline{y = x\cdot\ln x.}$$

Övning 2.4 (s.25)

$f(x,y,z) = \ln(x^3+y^3+z^3-3xyz);\quad x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z} = ?$

$f(x) = \ln u \wedge u = x^3+y^3+z^3-3xyz;$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = \dfrac{1}{u}\dfrac{\partial u}{\partial x} = \dfrac{3x^2-3yz}{u} \Rightarrow x\dfrac{\partial f}{\partial x} = \dfrac{3x^3-3xyz}{u} \\ \dfrac{\partial f}{\partial y} = \dfrac{1}{u}\dfrac{\partial f}{\partial y} = \dfrac{3y^2-3xz}{u} \Rightarrow y\dfrac{\partial f}{\partial y} = \dfrac{3y^3-3xyz}{u} \\ \dfrac{\partial f}{\partial z} = \dfrac{1}{u}\dfrac{\partial f}{\partial z} = \dfrac{3z^2-3xy}{u} \Rightarrow z\dfrac{\partial f}{\partial z} = \dfrac{3z^3-3xyz}{u} \end{array}\right\} \Rightarrow$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = \frac{3(x^3+y^3+z^3-3xyz)}{u} = \frac{3u}{u} = 3.$$

Övning 2.5 (s.25)

Se nästa sida.

$$\underline{Q(p,v) = g(pv) - f(pv)\ln p}$$

$$\begin{cases} g(pv) = g(t) \wedge t = pv \\ \dfrac{\partial g}{\partial p} = g'(t)\dfrac{\partial t}{\partial p} = g'(t)\cdot v = g'(pv)v \\ \dfrac{\partial g}{\partial v} = g'(t)\dfrac{\partial t}{\partial v} = g'(t)\cdot p = g'(pv)p \\ \\ f(pv) = f(t) \wedge t = pv \\ \dfrac{\partial f}{\partial p} = f'(t)\dfrac{\partial t}{\partial p} = f'(t)\cdot v = f'(pv)\cdot v \\ \dfrac{\partial f}{\partial v} = f'(t)\dfrac{\partial t}{\partial v} = f'(t)\cdot p = f'(pv)\cdot p \end{cases}$$

$$VL = v\frac{\partial Q}{\partial v} - p\frac{\partial Q}{\partial p} = v\left(\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\ln p\right) - p\left(\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\ln p - f\cdot\frac{1}{p}\right)$$

$$= v\left(g'(pv)p - f'(vp)p\ln p\right) - p\left(g'(pv)v - f'(pv)v\ln p\right) +$$

$$+ f(pv) = vp\left(g'(pv) - f'(pv)\ln p - g'(pv) + f'(pv)\ln p\right)$$

$$+ f(pv) = f(pv) = HL.$$

Övning 2.6 (s. 26)

a) $\dfrac{\partial f}{\partial x} = 2x\sin x^2,\ \dfrac{\partial f}{\partial y} = \cos y \quad (*)$

$$\frac{\partial f}{\partial x} = 2x\cdot\sin x^2 \Rightarrow f(x,y) = -\cos x^2 + \phi(y) \quad (**) \Rightarrow$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-\cos x^2 + \phi(y)\right) = \phi'(y) \overset{(*)}{=} \cos y \Leftrightarrow$$

$$\Leftrightarrow \phi(y) = \sin y + C \overset{(**)}{\Rightarrow} \underline{f(x,y) = -\cos x^2 + \sin y + C.}$$

b) $\frac{\partial f}{\partial x}=\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y}=-\frac{x}{x^2+y^2}$; (*)

$$\frac{\partial f}{\partial x}=\frac{y}{x^2+y^2}=\frac{y^2}{x^2+y^2}\cdot\frac{1}{y}=\frac{1}{(x/y)^2+1}\cdot\frac{\partial}{\partial x}\left(\frac{x}{y}\right)=$$

$$=\frac{\partial}{\partial x}\arctan\frac{x}{y}\Rightarrow f(x,y)=\arctan\frac{x}{y}+\phi(y) \quad (**)$$

$$\Rightarrow\frac{\partial f}{\partial y}=\frac{1}{(x/y)^2+1}\frac{\partial}{\partial y}\left(\frac{x}{y}\right)+\phi'(y)=\frac{y^2}{x^2+y^2}\left(-\frac{x}{y^2}\right)+\phi'(y)=$$

$$=-\frac{x}{x^2+y^2}+\phi'(y)\overset{(*)}{=}-\frac{x}{x^2+y^2}\Leftrightarrow\phi'(y)=0\Leftrightarrow\phi(y)=C$$

$$\overset{(**)}{\Rightarrow} f(x,y)=\arctan\frac{x}{y}+C.$$

Anm. $\arctan u+\arctan\frac{1}{u}=\frac{\pi}{2}\Leftrightarrow\arctan u=\frac{\pi}{2}-\arctan\frac{1}{u}$, så lösningen kan också skrivas som $f(x,y)=-\arctan\frac{y}{x}+C'$ $\left(C'=C+\frac{\pi}{2}\right)$.

Differentierbarhet

Jag använder Definition 2 på s. 43 i boken.

a) $f(x,y)=xy$, $P=(1,1)$.

$$f(1+h,1+k)-f(1,1)=(1+h)(1+k)-1=h+k+hk;$$

$$A_1=1=A_2,\quad \rho(h,k)=\frac{hk}{\sqrt{h^2+k^2}}.$$

$$|\rho(h,k)|=\frac{|hk|}{\sqrt{h^2+k^2}}\le\frac{\sqrt{h^2+k^2}\cdot\sqrt{h^2+k^2}}{\sqrt{h^2+k^2}}=\sqrt{h^2+k^2}\xrightarrow[(h,k)\to(0,0)]{}0$$

b) $f(x,y)=(1+x+2y)^2$, $P=(1,-1)$.

$$f(1+h,-1+k)-f(1,-1)=(h+2k)^2=0\cdot h+0\cdot k+(h+2k)^2;$$

$$A_1=0=A_2;\quad \rho(h,k)=\frac{(h+2k)^2}{\sqrt{h^2+k^2}};$$

$$|h+2k|\le|h|+2|k|\le\sqrt{h^2+k^2}+2\sqrt{h^2+k^2}=3\sqrt{h^2+k^2}\Rightarrow$$

$$\Rightarrow|\rho(h,k)|=\frac{|h+2k|^2}{\sqrt{h^2+k^2}}\le\frac{9(\sqrt{h^2+k^2})^2}{\sqrt{h^2+k^2}}=9\sqrt{h^2+k^2}\xrightarrow[\mathbf{h}\to 0]{}0.$$

c) $f(x,y)=e^{x+2y}$, $P=(2,2)$.

$$f(2+h,2+k)-f(2,2)=e^{6+h+2k}-e^6=e^6(e^{h+2k}-1)=$$

$$=e^6(e^h\cdot e^{2k}-1)=e^6((1+h+O(h^2))(1+k+O(k^2))-1)=$$

$$=e^6(1+h+k+O((\sqrt{h^2+k^2})^2)-1)=e^6(h+k+O(|\mathbf{h}|^2));$$

$$A_1=e^6=A_2,\quad \rho(\mathbf{h})=\frac{O(|\mathbf{h}|^2)}{|\mathbf{h}|}=O(|\mathbf{h}|)\xrightarrow[\mathbf{h}\to 0]{}0.$$

Anm. Om ordobeteckningen kan du läsa i A.4 på s. 374 i kursboken.

d) $f(x,y)=\sin(x+y)$, $P=(1,1)$.

$$f(1+h,1+k)-f(1,1)=\sin(2+(h+k))-\sin 2=$$

$$=\sin 2\cos(h+k)+\cos 2\sin(h+k)-\sin 2=$$

$$=\sin 2(\cos h\cos k-\sin h\sin k)+\cos 2(\sin h\cos k+$$

$$+\cos h\sin k)-\sin 2;$$

Vi sätter igång med att utveckla lite grann.

$$\cos h \cos k = (1+O(h^2))(1+O(k^2)) = 1+O((\sqrt{h^2+k^2})^2)$$

$$\sin h \sin k = (h+O(h^3))(k+O(k^3)) = hk + O((\sqrt{h^2+k^2})^4)$$
$$= O((\sqrt{h^2+k^2})^2).$$

$$\sin h \cos k = (h+O(h^3))(1+O(k^2)) = h + O((\sqrt{h^2+k^2})^2)$$

$$\cos h \sin k = (1+O(h^2))\cdot(k+O(k^3)) = k + O((\sqrt{h^2+k^2})^2)$$

$$f(1+h,1+k)-f(1,1) = \sin 2\,(1+O((\sqrt{h^2+k^2})^2)) + \cos 2\,(h+$$
$$+k+O((\sqrt{h^2+k^2})^2)) - \sin 2 = (\cos 2)h + (\cos 2)k + O(|\mathbf{h}|^2) \Rightarrow$$
$$\Rightarrow A_1 = \cos 2 = A_1' \;\wedge\; \rho(\mathbf{h}) = \frac{O(|\mathbf{h}|^2)}{|\mathbf{h}|} = O(|\mathbf{h}|) \xrightarrow[|\mathbf{h}|\to 0]{} 0.$$

Anm. I flera variabler är $h^2 = O(|\mathbf{h}|^2) = k^2$. Detsamma gäller hk och högre ordningens termer (monom).

Övning 2.8 (s. 26)

Samtliga är kontinuerligt deriverbara, dvs. ligger i C^1, så enligt sats 3 är de differentierbara.

Anm. Sammansättningar av de elementära funktionerna är differentierbara.

Övning 2.9 (s. 26)

$P = f(U,R) = \frac{U^2}{R}$; $U = 220\,V$, $R = 9\Omega$; $dU = 5$, $dR = 0{,}3$.

$$dP = \frac{\partial f}{\partial U}dU + \frac{\partial f}{\partial R}dR = \frac{2U}{R}dU - \frac{U^2}{R^2}dR;$$

$$dP = \frac{2}{9}\cdot 220\cdot 5 - \frac{220^2}{9^2}\cdot 0{,}3 = \underline{65\,W}.$$

Svar: Effekten ökar med 65 W.

Övning 2.10 (s. 26)

$$g = \frac{2s}{t^2} \Rightarrow dg = \frac{\partial g}{\partial s}ds + \frac{\partial g}{\partial t}dt = \frac{2}{t^2}ds - \frac{4s}{t^3}dt;$$

$\bar{s} = 2{,}0$, $ds = 0{,}01$; $\bar{t} = 0{,}63$, $dt = 0{,}01$.

$$dg = \frac{2}{0{,}63^2}\cdot 0{,}01 - \frac{4\cdot 2}{0{,}63^3}\,0{,}01 = -0{,}27 \;;\; \bar{g} = \frac{2\cdot 2{,}0}{0{,}63^2} = 10{,}10;$$

Resultat: $g = \bar{g} \pm |dg| = (10{,}10 \pm 0{,}27)\,m/s^2$.

Övning 2.11 (s. 26)

$z = x^2+4y^2$; $P = (1,1)$

$z = f(1,1) + f_x'(1,1)(x-1) + f_y'(1,1)(y-1)$ (Se (9) s. 45)

$$\frac{\partial f}{\partial x} = 2x \Rightarrow f_x'(1,1) = 2;\; \frac{\partial f}{\partial y} = 8y \Rightarrow f_y'(1,1) = 8.$$

$$z = 5 + 2(x-1) + 8(y-1) = 5+2x-2+8y-8 = 2x+8y-5 \Leftrightarrow$$
$$\Leftrightarrow \underline{2x+8y-z=5}.$$

Anm. Undvik beteckningen $\frac{\partial f}{\partial x}(1,1)$.

Övning 2.12 (s. 26)

$z = f(x,y) = x^2+4y^2$, $P=(a,b)$.

Tangentplanet har ekvationen

$$z = a^2+4b^2+f'_x(a,b)(x-a)+f'_y(a,b)(y-b);$$

$$z = a^2+4b+2a(x-a)+8b(y-b);$$

$$z = a^2+4b^2+2ax-2a^2+8by-8b^2 = -a^2-4b^2+2ax+8by.$$

$$-2ax-8by+z = -a^2-4b^2 \Leftrightarrow x+y+z=0 \quad (\text{Jfr VL});$$

$$-2a=1 \wedge -8b=1 \Leftrightarrow a=-\tfrac{1}{2} \wedge b=-\tfrac{1}{8} \Rightarrow a^2+4b^2=\tfrac{5}{16}.$$

Svar: $\left(-\frac{1}{2},-\frac{1}{8},\frac{5}{16}\right)$.

Kedjeregeln

Övning 2.13 (s. 26)

$f(x,y)=xy+e^{x^2y}$; $x=\cos t$, $y=\sin t$.

a) $u(t)=\cos t\sin t+e^{\cos^2 t\sin t}=\frac{1}{2}\sin 2t+e^{\cos^2 t\sin t}$

$u'(t)=\cos 2t+e^{\cos^2 t\sin t}(-2\cos t\sin^2 t+\cos^3 t)$

b) $\frac{\partial f}{\partial x}=y+2xye^{x^2y}$; $\frac{\partial f}{\partial y}=x+x^2e^{x^2y}$;

$$\begin{cases} f'_x(\cos t,\sin t)=\sin t+2\sin t\cos t\, e^{\cos^2 t\sin t} \\ f'_y(\cos t,\sin t)=\cos t+\cos^2 t\, e^{\cos^2 t\sin t} \end{cases}$$

$$\frac{du}{dt}=\frac{d}{dt}f(x(t),y(t))=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=$$

$$=f'_x(\cos t,\sin t)(-\sin t)+f'_y(\cos t,\sin t)\cos t=$$

$$=(\sin t+2\sin t\cos t\, e^{\cos^2 t\sin t})(-\sin t)+$$

$$+(\cos t+\cos^2 t\, e^{\cos^2 t\sin t})\cos t=$$

$$=-\sin^2 t-2\sin^2 t\cos t\, e^{\cos^2 t\sin t}+\cos^2 t+\cos^3 t\, e^{\cos^2 t\sin t}=$$

$$=\cos^2 t-\sin^2 t+e^{\cos^2 t\sin t}(-2\sin^2 t\cos t+\cos^3 t)=$$

$$=\underline{\cos 2t+e^{\cos^2 t\sin t}(-2\cos t\sin^2 t+\cos^3 t)}.$$

Övning 2.14 (s. 27)

Det gäller att visa att $f(x,y)=C$ för $3x+y=1$.

$3x+y=1 \Leftrightarrow y=1-3x. \Leftrightarrow x=t \wedge y=1-3t.$

$u(t)=f(t,1-3t)$; $x=t$, $y=1-3t$

$$u'(t)=\frac{du}{dt}=\frac{d}{dt}f(x(t),y(t))=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=$$

$$=f'_x(t,1-3t)\cdot 1+f'_y(t,1-3t)(-3)=\frac{\partial f}{\partial x}-3\frac{\partial f}{\partial y}=0\Rightarrow$$

$\Rightarrow u(t)=C \Leftrightarrow \underline{f(x,1-3x)=C}$ V.S.V.

Övning 2.15 (s. 27)

a) $u(x,y)=f(2x+3y)=f(v) \wedge v=2x+3y$;

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(v)=f'(v)\frac{\partial v}{\partial x}=f'(v)\cdot 2=2f'(2x+3y);$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}f(v)=f'(v)\frac{\partial v}{\partial y}=f'(v)\cdot 3=3f'(2x+3y);$$

$$VL=3\frac{\partial u}{\partial x}-2\frac{\partial u}{\partial y}=6f'(2x+3y)-6f'(2x+3y)=0=HL.$$

b) $u(x,y)=f(xy)=f(v)\ \wedge\ v=xy.$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(v)=f'(v)\frac{\partial v}{\partial x}=f'(v)y=yf'(xy);$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}f(v)=f'(v)\frac{\partial v}{\partial y}=f'(v)x=xf'(xy);$$

$$VL=x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=xyf'(xy)-xyf'(xy)=0=HL.$$

Övning 2.16 (s. 27)

$$2xy\frac{\partial u}{\partial x}-(2x+y^2)\frac{\partial u}{\partial y}=0.$$

a) $u(x,y)=f(x^2+xy^2)=f(v)\ \wedge\ v=x^2+xy^2;$

$$\frac{\partial v}{\partial x}=2x+y^2,\quad \frac{\partial v}{\partial y}=2xy;$$

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial}{\partial x}f(v)=f'(v)\dfrac{\partial v}{\partial x}=(2x+y^2)f'(x^2+xy^2);\\[2mm] \dfrac{\partial u}{\partial y}=\dfrac{\partial}{\partial y}f(v)=f'(v)\dfrac{\partial v}{\partial y}=2xyf'(x^2+xy^2);\end{cases}$$

$$VL=2xy\frac{\partial u}{\partial x}-(2x+y^2)\frac{\partial u}{\partial y}=2xy(2x+y^2)f'(x^2+xy^2)-$$
$$-(2x+y^2)\cdot 2xyf'(x^2+xy^2)=0=HL.$$

b) $u(x,0)=x\Rightarrow f(x^2)=x\Rightarrow f(t)=\sqrt{t}\Leftrightarrow f(v)=\sqrt{v}\Rightarrow$

$\Rightarrow u(x,y)=\sqrt{x^2+xy^2}.$

Anm: $f(x^2)=x>0\ \wedge\ t=x^2\Leftrightarrow f(t)=\sqrt{t}.$

Övning 2.17 (s. 27)

$$f(x,y)=\frac{1}{\sqrt{xy}}g\left(\frac{x}{y}\right)=\frac{1}{\sqrt{u}}g(v)\ \wedge\ u=xy\ \wedge\ v=\frac{x}{y};$$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\frac{1}{\sqrt{u}}g(v)=\left(\frac{\partial}{\partial u}\frac{1}{\sqrt{u}}g(v)\right)\frac{\partial u}{\partial x}+\left(\frac{\partial}{\partial v}\frac{1}{\sqrt{u}}g(v)\right)\frac{\partial v}{\partial x}=$$
$$=\left(-\frac{1}{2}u^{-3/2}g(v)\right)y+\left(\frac{1}{\sqrt{u}}g'(v)\right)\cdot\frac{1}{y}=$$
$$=-\frac{1}{2}(xy)^{-3/2}\cdot y\,g\left(\frac{x}{y}\right)+\frac{1}{y}\cdot\frac{1}{\sqrt{xy}}g'\left(\frac{x}{y}\right)=$$
$$=-\frac{1}{2}x^{-3/2}y^{-1/2}g\left(\frac{x}{y}\right)+x^{-1/2}y^{-3/2}g'\left(\frac{x}{y}\right).$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\frac{1}{\sqrt{u}}g(v)=\left(\frac{\partial}{\partial u}\frac{1}{\sqrt{u}}g(v)\right)\frac{\partial u}{\partial y}+\left(\frac{\partial}{\partial v}\frac{1}{\sqrt{u}}g(v)\right)\frac{\partial v}{\partial y}=$$
$$=\left(-\frac{1}{2}u^{-3/2}g(v)\right)x+\left(\frac{1}{\sqrt{u}}g'(v)\right)\left(-\frac{x}{y^2}\right)=$$
$$=-\frac{1}{2}(xy)^{-3/2}x\,g\left(\frac{x}{y}\right)-(xy)^{-1/2}xy^{-2}g'\left(\frac{x}{y}\right)=$$
$$=-\frac{1}{2}x^{-1/2}y^{-3/2}g\left(\frac{x}{y}\right)-x^{1/2}y^{-5/2}g'\left(\frac{x}{y}\right).$$

$$VL=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=-\frac{1}{2}x^{-1/2}y^{-1/2}g\left(\frac{x}{y}\right)+x^{1/2}y^{-3/2}g'\left(\frac{x}{y}\right)-$$
$$-\frac{1}{2}x^{-1/2}y^{-1/2}g\left(\frac{x}{y}\right)-x^{1/2}y^{-3/2}g'\left(\frac{x}{y}\right)+x^{-1/2}y^{-1/2}g\left(\frac{x}{y}\right)$$
$$=-x^{-1/2}y^{-1/2}g\left(\frac{x}{y}\right)+x^{-1/2}y^{-1/2}g\left(\frac{x}{y}\right)+x^{1/2}y^{-3/2}g'\left(\frac{x}{y}\right)-$$
$$-x^{1/2}y^{-3/2}g'\left(\frac{x}{y}\right)=0=HL.$$

Övning 2.18 (s. 27)

$$\begin{cases}u(x,y)=x+h(y)\Rightarrow\dfrac{\partial u}{\partial x}=1\ \wedge\ \dfrac{\partial u}{\partial y}=h'(y).\\[2mm] v(x,y)=y\Rightarrow\dfrac{\partial v}{\partial x}=0\ \wedge\ \dfrac{\partial v}{\partial y}=1.\end{cases}$$

forts.

$$f(x,y)=\tilde f(u,v)\Rightarrow\begin{cases}\dfrac{\partial f}{\partial x}=\dfrac{\partial \tilde f}{\partial x}=\dfrac{\partial \tilde f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial \tilde f}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{\partial \tilde f}{\partial u};\\[2mm] \dfrac{\partial f}{\partial y}=\dfrac{\partial \tilde f}{\partial y}=\dfrac{\partial \tilde f}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial \tilde f}{\partial v}\dfrac{\partial v}{\partial y}=h'(v)\dfrac{\partial \tilde f}{\partial u}+\dfrac{\partial \tilde f}{\partial v}\end{cases}$$

$$\Rightarrow x\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=(u-h(v))\frac{\partial \tilde f}{\partial u}+h'(v)\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}=$$

$$=(u-h(v)+h'(v))\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\equiv u\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\Leftrightarrow$$

$$\Leftrightarrow -h(v)+h'(v)=0\Leftrightarrow h(v)=Ce^{v}\Rightarrow \underline{h(y)=Ce^{y}}$$

Anm. $y=v$.

Övning 2.19 (s.28)

f differentierbar $\Rightarrow f_1'$ och f_2' kontinuerliga.

$$\begin{cases}u=\dfrac{x}{y}\Rightarrow \dfrac{\partial u}{\partial x}=\dfrac{1}{y}\ \wedge\ \dfrac{\partial u}{\partial y}=-\dfrac{x}{y^2}\ \wedge\ \dfrac{\partial u}{\partial z}=0;\\[2mm] v=\dfrac{y}{z}\Rightarrow \dfrac{\partial v}{\partial x}=0\ \wedge\ \dfrac{\partial v}{\partial y}=\dfrac{1}{z}\ \wedge\ \dfrac{\partial v}{\partial z}=-\dfrac{y}{z^2};\end{cases}$$

$$VL=x\frac{\partial h}{\partial x}+y\frac{\partial h}{\partial y}+z\frac{\partial h}{\partial z}=$$

$$=x\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)+z\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\right)$$

$$=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)\frac{\partial f}{\partial u}+\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}\right)\frac{\partial f}{\partial v}=$$

$$=\left(\frac{x}{y}-\frac{x}{y}+0\right)\frac{\partial f}{\partial u}+\left(0+\frac{y}{z}-\frac{y}{z}\right)\frac{\partial f}{\partial v}=0=HL.$$

Resultat: $x\dfrac{\partial h}{\partial x}+y\dfrac{\partial h}{\partial y}+z\dfrac{\partial h}{\partial z}=0.$

Övning 2.20 (s.28)

$f(x,y)=\tilde f(u,v)$; $u=ax+y,\ v=x.$

$$\frac{\partial f}{\partial x}-3\frac{\partial f}{\partial y}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}-3\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}\right)=$$

$$=\left(\frac{\partial u}{\partial x}-3\frac{\partial u}{\partial y}\right)\frac{\partial \tilde f}{\partial u}+\left(\frac{\partial v}{\partial x}-3\frac{\partial v}{\partial y}\right)\frac{\partial \tilde f}{\partial v}=(a-3)\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v};$$

$$\underline{a=3}\Rightarrow\frac{\partial \tilde f}{\partial v}=0\Leftrightarrow \tilde f(u,v)=\phi(u)\Rightarrow \underline{f(x,y)=\phi(3x+y)}.$$

Övning 2.21 (s.28)

$f(x,y)=\tilde f(u,v)$; $u=x-ky,\ v=x+ky.$

$$a)\begin{cases}\dfrac{\partial f}{\partial x}=\dfrac{\partial \tilde f}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial \tilde f}{\partial v}\dfrac{\partial v}{\partial x}=\dfrac{\partial \tilde f}{\partial u}\cdot 1+\dfrac{\partial \tilde f}{\partial v}\cdot 1=\dfrac{\partial \tilde f}{\partial u}+\dfrac{\partial \tilde f}{\partial v}\\[2mm] \dfrac{\partial f}{\partial y}=\dfrac{\partial \tilde f}{\partial u}\dfrac{\partial u}{\partial y}+\dfrac{\partial \tilde f}{\partial v}\dfrac{\partial v}{\partial y}=\dfrac{\partial \tilde f}{\partial u}(-k)+\dfrac{\partial \tilde f}{\partial v}(+k)=k\left(-\dfrac{\partial \tilde f}{\partial u}+\dfrac{\partial \tilde f}{\partial v}\right)\end{cases}$$

$$b)\ 2\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=(2-k)\frac{\partial \tilde f}{\partial u}+(2+k)\frac{\partial \tilde f}{\partial v};$$

$$\begin{cases}k=2\Rightarrow u=x-2y\ \wedge\ v=x+2y\Rightarrow 2\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=4\dfrac{\partial \tilde f}{\partial v};\\[2mm] k=-2\Rightarrow u=x+2y\ \wedge\ v=x-2y\Rightarrow 2\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=4\dfrac{\partial \tilde f}{\partial u};\end{cases}$$

(i) $\underline{k=2}$: $2\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=0\Rightarrow\dfrac{\partial \tilde f}{\partial v}=0\Leftrightarrow \tilde f(u,v)=\phi(u)\Leftrightarrow$

$\Leftrightarrow f(x,y)=\phi(x-2y)$;

$f(x,0)=\phi(x)=\sin x\Leftrightarrow \underline{\phi(u)=\sin u=\sin(x-2y)}.$

(ii) $\underline{k=-2}$: $2\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}=0\Rightarrow\dfrac{\partial \tilde f}{\partial u}=0\Leftrightarrow \tilde f(u,v)=\psi(v)\Leftrightarrow$

$\Leftrightarrow f(x,y)=\psi(x-2y)$;

$f(x,0)=\psi(x)=\sin x\Leftrightarrow \underline{\psi(v)=\sin v=\sin(x-2y)}.$

Resultat: $f(x,y)=\sin(x-2y)$ för $a=\pm 2$.

Övning 2.22 (s.28)

$f(x,y)=\tilde f(u,v)$; $u=xy$, $v=x/y$. (Jfr 2.17).

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=x\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}\right)+\tilde f$$

$$=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\frac{\partial \tilde f}{\partial u}+\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)\frac{\partial \tilde f}{\partial v}+\tilde f=$$

$$=(xy+xy)\frac{\partial \tilde f}{\partial u}+\left(\frac{x}{y}-\frac{x}{y}\right)\frac{\partial \tilde f}{\partial v}+\tilde f=$$

$$=2xy\frac{\partial \tilde f}{\partial u}+\tilde f=$$

$$=2u\frac{\partial \tilde f}{\partial u}+\tilde f;$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=0\Leftrightarrow 2u\frac{\partial \tilde f}{\partial u}+\tilde f=0\Leftrightarrow \frac{\partial \tilde f}{\partial u}+\frac{1}{2u}\tilde f=0;\ (*)$$

$$g(u)=\frac{1}{2u}\Rightarrow G(u)=\int_1^u g(\tau)\,d\tau=\ln\sqrt{u}\Rightarrow \mu(u)=\sqrt{u};$$

μ är en s.k. integrerande faktor till $(*)$.

$$\frac{\partial}{\partial u}(\sqrt{u}\,\tilde f)\equiv\sqrt{u}\frac{\partial \tilde f}{\partial u}+\frac{1}{2\sqrt{u}}\tilde f=0\Leftrightarrow \sqrt{u}\,\tilde f(u,v)=\phi(v)\Leftrightarrow$$

$$\Leftrightarrow \tilde f(u,v)=\frac{1}{\sqrt{u}}\phi(v)\Leftrightarrow \underline{f(x,y)=\frac{1}{\sqrt{xy}}\phi\left(\frac{x}{y}\right)}.$$

Övning 2.23 (s.28)

$$h(t)=\tilde f(tx,ty)\Rightarrow h'(t)=\frac{dh}{dt}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial t}=$$

$$=x\frac{\partial \tilde f}{\partial u}+y\frac{\partial \tilde f}{\partial v}\Rightarrow t\,h'(t)=tx\frac{\partial \tilde f}{\partial u}+ty\frac{\partial \tilde f}{\partial y}=$$

$$=u\frac{\partial \tilde f}{\partial u}+v\frac{\partial \tilde f}{\partial v}\Rightarrow t\,h'(t)+h(t)=u\frac{\partial \tilde f}{\partial u}+v\frac{\partial \tilde f}{\partial v}+\tilde f;$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+f=0\Rightarrow u\frac{\partial \tilde f}{\partial u}+v\frac{\partial \tilde f}{\partial v}+\tilde f=0\Rightarrow (t\,h(t))'=0$$

$$\Rightarrow t\cdot h(t)=C=0\cdot h(0)=0\Leftrightarrow h(t)\equiv 0\Rightarrow h(1)=0\Rightarrow$$

$$\Rightarrow f(x,y)\equiv 0,$$ V.S.V.

Övning 2.24 (s.29)

$f(x,y)=\tilde f(u,v)$; $x=u$, $y=\frac{u}{v}$; $\left(u=x,\ v=\frac{x}{y}\right)$.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=x\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}\right)=$$

$$=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\frac{\partial \tilde f}{\partial u}+\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right)\frac{\partial \tilde f}{\partial v}=$$

$$=(x\cdot 1+y\cdot 0)\frac{\partial \tilde f}{\partial u}+\left(x\cdot\frac{1}{y}+y\left(-\frac{x}{y^2}\right)\right)\frac{\partial \tilde f}{\partial v}=u\frac{\partial \tilde f}{\partial u};$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=y\Leftrightarrow u\frac{\partial \tilde f}{\partial u}=\frac{u}{v}\Leftrightarrow \frac{\partial \tilde f}{\partial u}=\frac{1}{v}\Rightarrow \tilde f(u,v)=$$

$$=\frac{u}{v}+\phi(v)\Leftrightarrow \underline{f(x,y)=y+\phi\left(\frac{x}{y}\right)}.$$

Övning 2.25 (s.29)

$f(x,y)=\tilde f(u,v)$; $u=x^2+y^2$, $v=e^{-x^2/2}$;

a)
$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=y\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}\right)-x\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}\right)=$$

$$=\left(y\frac{\partial u}{\partial x}-x\frac{\partial u}{\partial y}\right)\frac{\partial \tilde f}{\partial u}+\left(y\frac{\partial v}{\partial x}-x\frac{\partial v}{\partial y}\right)\frac{\partial \tilde f}{\partial v}=$$

$$=(y\cdot 2x-x\cdot 2y)\frac{\partial \tilde f}{\partial u}+\left(y(-x)e^{-x^2/2}-x\cdot 0\right)\frac{\partial \tilde f}{\partial v}=$$

$$=-xye^{-x^2/2}\frac{\partial \tilde f}{\partial v};$$

$$y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=xyf\Leftrightarrow -xye^{-x^2/2}\frac{\partial \tilde f}{\partial v}=xyf\Leftrightarrow$$

$$\Leftrightarrow -e^{-x^2/2}\frac{\partial \tilde f}{\partial v}=\tilde f\Leftrightarrow -v\frac{\partial \tilde f}{\partial v}=\tilde f\Leftrightarrow v\frac{\partial \tilde f}{\partial v}+f=0;$$

b) $v\frac{\partial\tilde f}{\partial v}+\tilde f=0 \Leftrightarrow \frac{\partial}{\partial v}(v\tilde f)=0 \Leftrightarrow v\tilde f(u,v)=\phi(u) \Leftrightarrow$

$\Leftrightarrow \tilde f(u,v)=\frac{1}{v}\phi(u) \Leftrightarrow f(x,y)=e^{x^2/2}\phi(x^2+y^2)$;

$f(0,y)=y^2 \Rightarrow \phi(y^2)=y^2 \Leftrightarrow \phi(u)=u=x^2+y^2$.

Resultat: $f(x,y)=(x^2+y^2)e^{x^2/2}$.

Övning 2.26 (s. 29)

$$\begin{cases} f(t)=F(t,-t) \Rightarrow f'(t)=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}=F'_x(t,-t)-F'_y(t,-t) \\ g(t)=F(t,2t) \Rightarrow g'(t)=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}=F'_x(t,2t)+2F'_y(t,2t) \end{cases}$$

$$\begin{cases} f'(0)=2 \Rightarrow F'_x(0,0)-F'_y(0,0)=2 \\ g'(0)=0 \Rightarrow F'_x(0,0)+2F'_y(0,0)=0 \end{cases} \Leftrightarrow \begin{cases} F'_x(0,0)=\frac{4}{3} \\ F'_y(0,0)=-\frac{2}{3} \end{cases}.$$

Övning 2.27 (s. 29)

$$\frac{\partial u}{\partial t_1}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1} \wedge \frac{\partial u}{\partial t_2}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_2}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_2} \Rightarrow$$

$$\Rightarrow \begin{cases} u'_1(t_1,t_2)=f'_1(x_1,x_2)g'_1(t_1,t_2)+f'_2(x_1,x_2)h'_1(t_1,t_2) \\ u'_2(t_1,t_2)=f'_1(x_1,x_2)g'_2(t_1,t_2)+f'_2(x_1,x_2)h'_2(t_1,t_2) \end{cases} ;(*)$$

$$(t_1,t_2)=(1,1) \Rightarrow \begin{cases} g(1,1)=1 \\ h(1,1)=2 \end{cases} \Rightarrow \begin{cases} f'_1(1,2)=7 \\ f'_2(1,2)=-4 \end{cases};$$

$$(*) \Rightarrow \begin{cases} u'_1(1,1)=f'_1(1,2)g'_1(1,1)+f'_2(1,2)h'_1(1,1)=7\cdot 2-4\cdot 1=10 \\ u'_2(1,1)=f'_1(1,2)g'_2(1,1)+f'_2(1,2)h'_2(1,1)=7\cdot 1-4\cdot 2=-1 \end{cases}$$

Gradient och riktningsderivata

Övning 2.28 (s. 29)

a) $f(x,y)=(x^2+y^2)^n=u^n,\ u=x^2+y^2$.

$$\begin{cases} \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}u^n=nu^{n-1}\frac{\partial u}{\partial x}=2nx\cdot(x^2+y^2)^{n-1}; \\ \frac{\partial f}{\partial y}=\frac{\partial}{\partial y}u^n=nu^{n-1}\frac{\partial u}{\partial y}=2ny(x^2+y^2)^{n-1}; \end{cases}$$

$\mathrm{grad}((x^2+y^2)^n)=(2nx(x^2+y^2)^{n-1}, 2ny(x^2+y^2)^{n-1})$.

b) $f(x,y,z)=e^{xyz}=e^u \wedge u=xyz$

$$\begin{cases} \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}e^u=e^u\frac{\partial u}{\partial x}=e^u yz=yze^{xyz}. \\ \frac{\partial f}{\partial y}=\frac{\partial}{\partial y}e^u=e^u\frac{\partial u}{\partial y}=e^u xz=xze^{xyz}. \\ \frac{\partial f}{\partial z}=\frac{\partial}{\partial z}e^u=e^u\frac{\partial u}{\partial z}=e^u xy=xye^{xyz}. \end{cases}$$

$\mathrm{grad}(e^{xyz})=(yze^{xyz}, xze^{xyz}, xye^{xyz})$.

c) $f(x,y)=\ln((x-a)^2+(y-b)^2)=\ln u \wedge u=(x-a)^2+(y-b)^2$

$$\begin{cases} \frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\ln u=\frac{1}{u}\frac{\partial u}{\partial x}=\frac{2(x-a)}{(x-a)^2+(y-b)^2} \\ \frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\ln u=\frac{1}{u}\frac{\partial u}{\partial y}=\frac{2(y-b)}{(x-a)^2+(y-b)^2} \end{cases}$$

$\mathrm{grad}(\ln((x-a)^2+(y-b)^2))=\left(\frac{2(x-a)}{(x-a)^2+(y-b)^2}, \frac{2(y-b)}{(x-a)^2+(y-b)^2}\right)$.

d) $f(\mathbf{x})=|\mathbf{x}|,\ \mathbf{x}=(x_1,x_2,x_3,\dots,x_n)$

$f(\mathbf{x})=\sqrt{u} \wedge u=x_1^2+x_2^2+x_3^2+\dots+x_n^2$.

$$f_i'(\mathbf{x}) = \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{\partial u}{\partial x_i} = \frac{2x_i}{2\sqrt{u}} = \frac{x_i}{|\mathbf{x}|}, \quad i=1,2,\dots,n.$$

$$\text{grad}\, f(\mathbf{x}) = \left(\frac{x_1}{|\mathbf{x}|}, \frac{x_2}{|\mathbf{x}|}, \dots, \frac{x_n}{|\mathbf{x}|}\right) = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Övning 2.29 (s. 30)

$$F(x,y) = \frac{1}{\sqrt{x^2+y^2}} = u^{-1/2} \ \wedge \ u = x^2+y^2.$$

$$\begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{\partial}{\partial x} u^{-1/2} = -\dfrac{1}{2}u^{-3/2}\dfrac{\partial u}{\partial x} = -\dfrac{1}{2}u^{-3/2}\cdot 2x = \dfrac{-x}{|\mathbf{x}|^3} ; \\ \dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y} u^{-1/2} = -\dfrac{1}{2}u^{-3/2}\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}u^{-3/2}\cdot 2y = \dfrac{-y}{|\mathbf{x}|^3} ; \end{cases}$$

$$\text{grad}\left(\frac{1}{r}\right) = \text{grad}\, F(x,y) = \left(-\frac{x}{|\mathbf{x}|^3}, -\frac{y}{|\mathbf{x}|^3}\right) = -\frac{1}{|\mathbf{x}|^3}(x,y) = -\frac{1}{r^3}\mathbf{r}.$$

Övning 2.30 (s. 30)

$$f(x,y,z) = \frac{xy^2z^3}{x+2} = \frac{x}{x+2}y^2z^3. \qquad \mathbf{v} = (-4,2,-4)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\dfrac{x}{x+2}y^2z^3 = \dfrac{2y^2z^3}{(x+2)^2} ; \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\dfrac{xy^2z^3}{x+2} = \dfrac{2xyz^3}{(x+2)^2} \\ \dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\dfrac{xy^2z^3}{x+2} = \dfrac{3xy^2z^2}{x+2} \end{cases} \Rightarrow \text{grad}\, f(x,y,z) =$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{2y^2z^3}{(x+2)^2}, \frac{2xyz^3}{x+2}, \frac{3xy^2z^2}{x+2}\right) \Rightarrow$$

$$\Rightarrow \text{grad}\, f(2,2,1) = \left(\tfrac{1}{2}, 2, 6\right).$$

$$\mathbf{v} = (-4,2,-4) \Rightarrow |\mathbf{v}| = \sqrt{4^2+2^2+4^2} = 6 \Rightarrow \hat{\mathbf{v}} = \left(-\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right);$$

$$\frac{\partial f}{\partial \mathbf{v}} = \text{grad}\, f(2,2,1)\cdot\hat{\mathbf{v}} = \left(\tfrac{1}{2}, 2, 6\right)\cdot\left(-\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right) = -\frac{11}{3}.$$

Anm. $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ enhetsvektor i $\mathbf{v}$:s riktning.

Övning 2.31 (s. 30)

$$f(x,y,z) = xyz \ ; \quad \mathbf{v} = (1,2,2)$$

$$\text{grad}\, f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (yz, xz, xy)$$

$$\mathbf{v} = (1,2,2) \Rightarrow |\mathbf{v}| = \sqrt{1^2+2^2+2^2} = 3 \Rightarrow \hat{\mathbf{v}} = \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right);$$

$$\frac{\partial f}{\partial \mathbf{v}} = \text{grad}\, f(x,y,z)\cdot\hat{\mathbf{v}} = (yz, xz, xy)\cdot\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right) \Leftrightarrow$$

$$f_{\mathbf{v}}'(x,y,z) = \tfrac{1}{3}yz + \tfrac{2}{3}xz + \tfrac{2}{3}xy.$$

Övning 2.32 (s. 30)

$$f(x,y,z) = \frac{(x^2+y^2)z}{2-z^2} ; \quad A = (-1,2,1), \ B = (0,4,-1)$$

$$\mathbf{v} = \overline{AB} = \overline{OB} - \overline{OA} = (0,4,-1)_{\bar{e}} - (-1,2,1)_{\bar{e}} = (1,2,-2) \Rightarrow$$

$$\Rightarrow |\mathbf{v}| = \sqrt{1^2+2^2+2^2} = 3 \Rightarrow \hat{\mathbf{v}} = \left(\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right); \ (*)$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{2xz}{2-z^2} \\ \dfrac{\partial f}{\partial y} = \dfrac{2yz}{2-z^2} \\ \dfrac{\partial f}{\partial z} = \dfrac{(x^2+y^2)(2+z^2)}{(2-z^2)^2} \end{cases} \Rightarrow \text{grad}\, f(-1,2,1) = (-2,4,15) \overset{(*)}{\Rightarrow}$$

$$\Rightarrow \frac{\partial f}{\partial \mathbf{v}} = \text{grad}\, f(-1,2,1)\cdot\hat{\mathbf{v}} = (-2,4,15)\cdot\left(\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right) = -8.$$

Övning 2.33 (s. 30)

$$f(x,y,z) = x^2+y^2+z^2, \quad P = (2,3,6).$$

$$\text{grad}\, f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z);$$

Låt $\hat{v}$ vara en enhetsriktning, dvs. $|\hat{v}|=1$.

$\frac{\partial f}{\partial v} = \operatorname{grad} f(2,3,6)\cdot\hat{v} = (4,6,12)\cdot\hat{v} = |(4,6,12)|\cos\theta$

θ är vinkeln mellan gradienten och $\hat{v}$.

$\left|\frac{\partial f}{\partial v}\right| = 14|\cos\theta| \leq 14 \Leftrightarrow \underline{-14 \leq \frac{\partial f}{\partial v} \leq 14}$.

Resultat: Riktningsderivatan antar alla värden i intervallet $[-14,14]$.

Övning 2.34 (s. 30)

$\underline{F(x,y,z) = xy + e^{yz} + z\,,\quad \hat{v} = (\alpha,\beta,\gamma)}$.

$\operatorname{grad}F(x,y,z) = \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right) = (y,\, x+ze^{yz},\, ye^{zy}+1) \Rightarrow$

$\Rightarrow \frac{\partial F}{\partial v} = \operatorname{grad}F(x,y,z)\cdot\hat{v} = \alpha y + \beta(x+ze^{yz}) + \gamma(ye^{yz}+1)$.

$\frac{\partial F}{\partial v}$ blir maximal när $\hat{v}$ är parallell med gradienten, dvs. när $\hat{v} = \frac{1}{|\operatorname{grad}F|}\operatorname{grad}F$. (s. 67).

Övning 2.35 (s. 30)

Myran ska börja krypa i gradientens riktning.

$T(x,y) = \frac{20}{\pi}\arctan\left(\frac{2\cos x}{e^{y}-e^{-y}}\right) = \frac{20}{\pi}\arctan\frac{\cos x}{\sinh y}$

$T(x,y) = \frac{20}{\pi}\arctan u \;\wedge\; u = \frac{\cos x}{\sinh y}$;

$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\frac{20}{\pi}\arctan u = \frac{20}{\pi}\frac{1}{1+u^2}\frac{\partial u}{\partial x} = \frac{20}{\pi}\frac{1}{1+u^2}\left(-\frac{\sin x}{\sinh y}\right)$.

$\frac{\partial T}{\partial y} = \frac{20}{\pi}\frac{1}{1+u^2}\frac{\partial u}{\partial y} = \frac{20}{\pi}\frac{1}{1+u^2}\left(-\frac{\cos x\cosh y}{\sinh^2 y}\right)$

$$\begin{cases} u(x,y) = \frac{\cos x}{\sinh y} \Rightarrow u\left(\frac{\pi}{3},\ln 2\right) = \frac{1/2}{3/4} = \frac{2}{3} \Rightarrow \frac{1}{1+u^2} = \frac{9}{13}\,; \\ v(x,y) = -\frac{\sin x}{\sinh y} \Rightarrow v\left(\frac{\pi}{3},\ln 2\right) = -\frac{\sqrt{3}/2}{3/4} = -\frac{2\sqrt{3}}{3}\,; \\ w(x,y) = -\frac{\cos x\cosh y}{\sinh^2 y} \Rightarrow w\left(\frac{\pi}{3},\ln 2\right) = -\frac{1/2\cdot 5/4}{9/16} = -\frac{10}{9}\,; \end{cases}$$

$\operatorname{grad}T\left(\frac{\pi}{3},\ln 2\right) = \frac{20}{\pi}\cdot\frac{9}{13}\left(-\frac{2\sqrt{3}}{3},-\frac{10}{9}\right) = \frac{40\pi}{13}(-3\sqrt{3},-5)$.

Svar: Myran ska börja krypa i riktningen $(-3\sqrt{3},-5)$ för att komma in i värmen igen.

Övning 2.36 (s. 30)

$z = f(x,y) = \frac{32}{1+x^2+y^2} \Rightarrow \operatorname{grad} f(x,y) = -\frac{64}{(1+x^2+y^2)^2}(x,y)$

$\Rightarrow |\operatorname{grad} f(x,y)| = 64\frac{\sqrt{x^2+y^2}}{(1+x^2+y^2)^2} = \phi(\sqrt{x^2+y^2})$;

$\phi(r) = \frac{64r}{(1+r^2)^2} \Rightarrow \phi'(r) = \frac{64-192r^2}{(1+r^2)^3}$;

$\phi'(r) = 0 \Rightarrow 64-192r^2 = 0 \Leftrightarrow r = \frac{1}{\sqrt{3}}$ (cirkel) $\Rightarrow$

$\Rightarrow \phi\left(\frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{3}}\cos\theta, \frac{1}{\sqrt{3}}\sin\theta\right) = \frac{32}{1+1/3} = 24$.

Svar: Kullen stiger brantast på höjden 24m.

Övning 2.37 (s. 31)

$$\begin{cases} x^2-y^2=3 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x^2-y^2=3 \\ 2xy=4 \end{cases} \Leftrightarrow \begin{cases} x^2-y^2=3 \\ x^2+y^2=5 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x^2=4 \\ y^2=1 \\ xy=2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x = \pm 2 \\ y = \pm 1 \\ xy = 2 \end{cases} \Leftrightarrow \underline{(x,y) = (2,1)} \vee \underline{(x,y) = (-2,-1)}.$$

<u>Anm</u>. $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$x^2-y^2=3$ är nivåkurva till $f(x,y) = x^2-y^2$ och $xy=2$ är nivåkurva till $g(x,y) = xy$.

Skärningsvinkeln θ fås ur ekvationen

$$(*) \qquad \text{grad} f \cdot \text{grad} g = |\text{grad} f| \cdot |\text{grad} g| \cos\theta.$$

Observera att vinkeln mellan tangenterna är lika med vinkeln mellan normalerna i samma punkt. Det är alltid den spetsiga vinkeln som avses (om den inte är rät, förstås).

$$\begin{cases} \text{grad} f(x,y) = (2x, -2y) \\ \text{grad} g(x,y) = (y, x) \end{cases} \overset{(*)}{\Rightarrow} \text{grad} f \cdot \text{grad} g = 0.$$

Skärningsvinklarna är räta i varje punkt och följaktligen i skärningspunkterna $(2,1), (-2,-1)$.

<u>Övning 2.38 (s. 31)</u>

$\underline{f(x,y) = 5x^2+5xy+3y^2-8x-6y+3}$ (Jfr. 1.24).

a) $f(t^2, t+1) = 5t^3+5t^4 = 0 \Leftrightarrow t=0 \vee t=-1 \Rightarrow$

$$\Rightarrow \underline{(x(0), y(0)) = (0,1)} \vee \underline{(x(-1), y(-1)) = (1,0)}$$

b) $y = t+1 \Leftrightarrow t = y-1 \Rightarrow t^2 = x = (y-1)^2 \Leftrightarrow x-(y-1)^2 = 0.$

$x-(y-1)^2=0$ är nivåkurva till $g(x,y) = x-(y-1)^2$

$$\begin{cases} \text{grad} f(x,y) = (10x+5y-8, 5x+6y-6) \\ \text{grad} g(x,y) = (1, 2-2y) \end{cases};$$

$\text{grad} f(0,1) \cdot \text{grad} g(0,1) = (-3,0)\cdot(1,0) = 3\cos\theta \Leftrightarrow$

$\Leftrightarrow 3 = 3\cos\theta \Leftrightarrow \underline{\theta = 0}.$

$\text{grad} f(1,0) \cdot \text{grad} g(1,0) = (2,-1)\cdot(1,2) = 0 \Rightarrow \theta = 90°.$

<u>Resultat</u>: a) Kurvorna skär varandra i punkterna $(0,1)$ och $(1,0)$.

b) I $(0,1)$ tangerar kurvorna varandra, dvs. skärningsvinkeln där är $0°$; i $(1,0)$ skär kurvorna varandra under rät vinkel.

<u>Övning 2.39 (s. 31)</u>

$$\begin{cases} 4x^3+4y^2 = 13 \\ 4y^2-4x^3 = 5 \end{cases} \Leftrightarrow \begin{cases} x^3 = 1 \\ y^2 = \frac{9}{4} \end{cases} \Leftrightarrow \underline{(x,y) = (1, \tfrac{3}{2})} \vee \underline{(x,y) = (1, -\tfrac{3}{2})}.$$

$4x^3+4y^2=13$ är nivåkurva till $f(x,y)=4x^3+4y^2$.

$4x^3-4y^2=-5$ är nivåkurva till $g(x,y)=4x^3-4y^2$.

$$\begin{cases} f(x,y)=4x^3+4y^2 \Rightarrow \operatorname{grad} f(x,y)=(12x^2, 8y). \\ g(x,y)=4x^3-4y^2 \Rightarrow \operatorname{grad} g(x,y)=(12x^2,-8y). \end{cases}$$

$\operatorname{grad} f(1,\frac{3}{2})\cdot \operatorname{grad} g(1,\frac{3}{2})=(12,12)\cdot(12,-12)=0 \Rightarrow \theta=90°$.

$\operatorname{grad} f(1,-\frac{3}{2})\cdot \operatorname{grad} g(1,-\frac{3}{2})=(12,-12)\cdot(12,12)=0 \Rightarrow \theta=90°$.

Resultat: Kurvorna skär varandra under rät vinkel i punkterna $(1,\frac{3}{2}),(1,-\frac{3}{2})$.

Övning 2.40 (s. 31)

a) $f(x,y,z)=x^2z-2xy-y^2+z \Rightarrow f(0,-1,1)=0$

$\operatorname{grad} f(x,y,z)=(2xz-2y,-2x-2y,x^2+1)$

$\operatorname{grad} f(0,-1,1)=(2,2,1)$

$\pi: \operatorname{grad} f(0,-1,1)\cdot(x-0,y+1,z-1)=0 \Leftrightarrow$

$\Leftrightarrow (2,2,1)\cdot(x,y+1,z-1)=0 \Leftrightarrow 2x+2y+z+1=0.$

b) $f(x,y,z)=z-e^{xz+2y} \Rightarrow f(0,0,1)=0.$

$\operatorname{grad} f(x,y,z)=(-ze^{xz+2y},-2e^{xz+2y},1-xe^{xz+2y})$

$\operatorname{grad} f(0,0,1)=(-1,-2,1)=-(1,2,-1);$

forts.

$\pi: \operatorname{grad} f(0,0,1)\cdot(x,y,z-1)=0 \Leftrightarrow (1,2,-1)\cdot(x,y,z-1)=0$

$\Leftrightarrow x+2y-z+1=0$

c) $f(x,y,z)=xyz-\arctan(x+y+z) \Rightarrow f(1,-1,0)=0.$

$$\operatorname{grad} f(x)=\left(yz-\frac{1}{1+(x+y+z)^2}, xz-\frac{1}{1+(x+y+z)^2}, xy-\frac{1}{1+(x+y+z)^2}\right).$$

$\operatorname{grad} f(1,-1,0)=(0-1,0-1,-1-1)=(-1,-1,-2)=-(1,1,2).$

$\pi: \operatorname{grad} f(1,-1,0)\cdot(x-1,y+1,z)=0 \Leftrightarrow (1,1,2)\cdot(x-1,y+1,z)$

$=x-1+y+1+2z=0 \Leftrightarrow x+y+2z=0.$

Övning 2.41 (s. 31)

a) $f(x,y,z)=xz^5+xyz-9 \Rightarrow f(3,2,1)=3+6-9=0.$

$\operatorname{grad} f(x,y,z)=(z^5+yz,xz,5xz^4+xy);$

$\operatorname{grad} f(3,2,1)=(1+2,3,15+6)=3(1,1,7);$

$\pi: \operatorname{grad} f(3,2,1)\cdot(x-1,y-1,z-3)=0 \Leftrightarrow (1,1,7)\cdot(x-3,$

$y-2,z-1)=0 \Leftrightarrow x-3+y-2+7z-7=0 \Leftrightarrow x+y+7z=12.$

b) $x+y+7z=12 \Leftrightarrow 7z=12-x-y \Leftrightarrow z=\frac{12-x-y}{7};$

Den sökta approximationen är

$$z=\frac{12-3{,}4-2{,}3}{7}=\frac{6{,}3}{7}=0{,}9.$$

Övning 2.42 (s.31)

$n=(1,1,1)$ är en normalvektor till planet.

$x^2+y^2-z^2=1$ är en nivåyta till funktionen

$$f(x,y,z)=x^2+y^2-z^2.$$

Vi kallar tangeringspunkten (a,b,c). I denna punkt har vi $\text{grad} f(a,b,c) \parallel n$.

$$(2a,2b,-2c)=k\cdot(1,1,1) \Leftrightarrow$$

$$\Leftrightarrow 2a=2b=-2c=k \Leftrightarrow a=b=-c=\frac{k}{2};$$

$f(a,b,c)=1 \Rightarrow \frac{k^2}{4}+\frac{k^2}{4}-\frac{k^2}{4}=1 \Leftrightarrow k^2=4 \Leftrightarrow \underline{k=-2} \vee \underline{k=2}.$

(i) $\underline{k=2} \Rightarrow a=b=-c=1 \Rightarrow d=1+1-1=1.$

(ii) $\underline{k=-2} \Rightarrow a=b=-c=-1 \Rightarrow d=-1-1+1=-1.$

Övning 2.43 (s.31)

$x^2-y^2+z^2+2=0$ är en nivåyta till funktionen $f(x,y,z)=x^2-y^2+z^2$ och $x^2+y^2+3z^2=8$ en nivåyta till funktionen $g(x,y,z)=x^2+y^2+3z^2$.

Låt (a,b,c) vara en sådan punkt.

$\underline{\text{grad} f(a,b,c)\cdot \text{grad} g(a,b,c)=0} \Leftrightarrow (2a,-2b,2c)\cdot(2a,2b,6c)=0 \Leftrightarrow (a,-b,c)\cdot(a,b,3c)=a^2-b^2+3c^2=0$

$\Leftrightarrow 3c^2=b^2-a^2.$ (*)

$\underline{g(a,b,c)=8} \Rightarrow a^2+b^2+3c^2=8 \overset{(*)}{\Rightarrow} a^2+b^2+b^2-a^2=8 \Leftrightarrow$

$\Leftrightarrow 2b^2=8 \Leftrightarrow b^2=4 \Leftrightarrow \underline{b=2}$ (ty $b>0$).

$\underline{f(a,b,c)=-2} \Rightarrow a^2-b^2+c^2=-2 \Leftrightarrow c^2=-2+b^2-a^2 \overset{(*)}{=} -2+3c^2$

$\Leftrightarrow 2c^2=2 \Leftrightarrow c^2=1 \Leftrightarrow \underline{c=1}$ $(c>0)$.

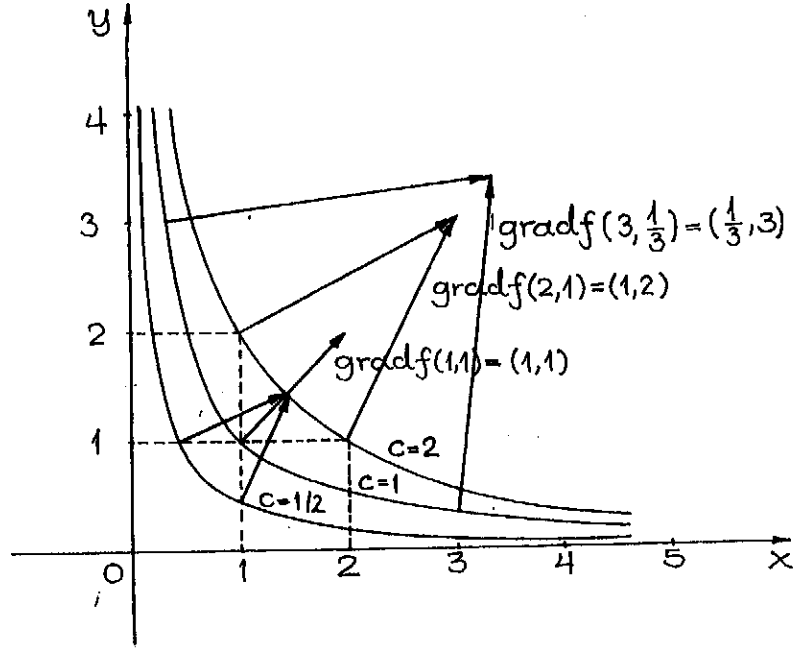
(*) $a^2=b^2-3c^2=1 \Leftrightarrow \underline{a=1}.$

<u>Resultat</u>: $(1,2,1)$ är den enda punkten med de givna egenskaperna.

Övning 2.44 (s.31)

a) $f(x,y)=xy \Rightarrow \text{grad} f(x,y)=(y,x).$

$\text{grad} f(a,b)=(b,a)$ är en vektor med fotpunkten på (a,b) på kurvan. Jag betraktar restriktionen till den första kvadranten, dvs. för $a>0, b>0$. I facit kan du finna hela porträttet. $|\text{grad} f(a,b)|$ är f: maximala tillväxthastighet i punkten (a,b).



b) $f(x,y) = x^2y^3 \Rightarrow \frac{\partial f}{\partial x} = 2xy^3 \wedge \frac{\partial f}{\partial y} = 3x^2y^2.$

$z = f(2,1) + f'_x(2,1)(x-2) + f'_y(2,1)(y-1)$

$z = 4 + 4(x-2) + 12(y-1) \Leftrightarrow \underline{4x+12y-z=16.}$

Övning 2.45 (s. 31)

$z = f(x,y)$ är en nivåyta till $\underline{F(x,y,z) = f(x,y) - z}.$

$\operatorname{grad} F(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right).$

$\operatorname{grad} F(0,0,0) = (f'_x(0,0), f'_y(0,0), -1) = k(1,1,1). \Leftrightarrow$

$\Leftrightarrow f'_x(0,0) = -1 = f'_y(0,0).$ (*)

$z = g(x,y) = (2+f(x,y))^2 \Rightarrow g(0,0) = 2^2 = 4.$

$\frac{\partial g}{\partial x} = 2(2+f(x,y))\frac{\partial f}{\partial x}, \quad \frac{\partial g}{\partial y} = 2(2+f(x,y))\frac{\partial f}{\partial y};$

$g'_x(0,0) = 2(2+f(0,0))f'_x(0,0) = 2(2+0)(-1) = -4;$

$g'_y(0,0) = 2(2+f(0,0))f'_y(0,0) = 2(2+0)(-1) = -4;$

$z = g(0,0) + g'_x(0,0)x + g'_y(0,0)y \Leftrightarrow z = 4-4x-4y \Leftrightarrow$

$\Leftrightarrow \underline{4x+4y+z-4=0.}$

Övning 2.46 (s. 32)

Ellipsoiden $2x^2+y^2+z^2=4$ är en nivåyta till funktionen $f(x,y,z) = 2x^2+y^2+z^2.$

$\operatorname{grad} f(x,y,z) = (4x, 2y, 2z) \Rightarrow \operatorname{grad} f(1,1,1) = (4,2,2)$

$n = (1,-1,2)$ är en normalvektor till planet $x-y+2z=2$, så tangentriktningen blir

$$n \times \operatorname{grad} f(1,1,1) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 4 & 2 & 2 \\ 1 & -1 & 2 \end{vmatrix} = (6,-6,-6) = -6(-1,1,1).$$

Den sökta tangentriktningen är $(-1,1,1)$.

Övning 2.47 (s. 32)

a) $f(t\mathbf{x}) = t f(\mathbf{x}) \Rightarrow \frac{d}{dt} f(tx,ty) = \frac{d}{dt} t f(x,y) = f(x,y)$

Kedjeregeln med $u=tx$, $v=ty$ leder till

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}=x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial y}=(x,y)\operatorname{grad} f(t\mathbf{x})=f(\mathbf{x})$$

$$t=1\Rightarrow (x,y)\operatorname{grad} f(\mathbf{x})=f(\mathbf{x})\Leftrightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f.$$

b) $F(x,y,z)=f(x,y)-z \Rightarrow \operatorname{grad} F(\mathbf{x})=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1)$.

Ur a) fås $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=f(\mathbf{x})=z$ s.a. i (ξ,η,ζ) fås

$$xf'_x(\xi,\eta)+yf'_y(\xi,\eta)-z=0.$$

Detta är ekvationen för ett plan gm origo.

Övning 2.48 (s. 32)

$$\operatorname{grad} F(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=k\cdot(x,y,z)\Leftrightarrow\begin{cases}\frac{\partial F}{\partial x}=kx\\ \frac{\partial F}{\partial y}=ky\\ \frac{\partial F}{\partial z}=kz\end{cases}\quad(*)$$

$$\frac{\partial F}{\partial x}=kx\Leftrightarrow F(x,y,z)=\frac{1}{2}kx^2+\phi(y,z)$$

$$\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}(\frac{1}{2}kx^2+\phi(y,z))=\frac{\partial\phi}{\partial y}\overset{(*)}{=}ky\Rightarrow\phi(y,z)=\frac{1}{2}ky^2+\psi(z)$$

$$\Rightarrow F(x,y,z)=\frac{1}{2}kx^2+\frac{1}{2}ky^2+\psi(z)\Rightarrow\frac{\partial F}{\partial z}=\psi'(z)\overset{(*)}{=}kz\Leftrightarrow$$

$$\Leftrightarrow\psi(z)=\frac{1}{2}kz^2+C\Rightarrow F(x,y,z)=\frac{1}{2}kx^2+\frac{1}{2}ky^2+\frac{1}{2}kz^2.$$

De sökta ytorna är nivåytor till $F(x,y,z)$, s.a.

$$\frac{1}{2}kx^2+\frac{1}{2}ky^2+\frac{1}{2}kz^2=\frac{1}{2}kR^2\Leftrightarrow x^2+y^2+z^2=R^2.$$

<u>Resultat</u>: De sökta funktionsytorna är halvsfärer, $z=\sqrt{R^2-x^2-y^2}$ eller $z=-\sqrt{R^2-x^2-y^2}$.

<u>Anm.</u> En sfär är ingen funktionsyta.

Partiella derivator av högre ordning

Övning 2.50 (s. 32)

<u>$f(x,y,z)=\sin(x^2+y^2)+xyz$</u>

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\sin(x^2+y^2)+\frac{\partial}{\partial x}xyz=\cos(x^2+y^2)\frac{\partial}{\partial x}(x^2+y^2)+yz=$$
$$=2x\cdot\cos(x^2+y^2)+yz;$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\sin(x^2+y^2)+\frac{\partial}{\partial y}xyz=2y\cdot\cos(x^2+y^2)+xz;$$

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\sin(x^2+y^2)+\frac{\partial}{\partial z}xyz=xy.$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(2x\cos(x^2+y^2)+yz)=\frac{\partial}{\partial x}2x\cos(x^2+y^2)=$$
$$=2\cos(x^2+y^2)+2x(-2x\sin(x^2+y^2))=$$
$$=2\cos(x^2+y^2)-4x^2\sin(x^2+y^2).$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(2y\cos(x^2+y^2)+xz)=\frac{\partial}{\partial y}2y\cos(x^2+y^2)=$$
$$=2\cos(x^2+y^2)-4y^2\sin(x^2+y^2).$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial x}(2y\cos(x^2+y^2)+xz)=$$
$$=2y(-2x\sin(x^2+y^2))+z=-4xy\sin(x^2+y^2)+z.$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial y}(2x\cos(x^2+y^2)+yz)=$$
$$=2x(-2y\sin(x^2+y^2))+z=-4xy\sin(x^2+y^2)+z.$$

$$\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial z})=\frac{\partial}{\partial x}(xy)=y.$$

$$\frac{\partial^2 f}{\partial z\partial x}=\frac{\partial}{\partial z}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial z}(2x\cos(x^2+y^2)+yz)=y.$$

$$\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial y}(xy)=x.$$

$$\frac{\partial^2 f}{\partial z\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial z}\left(2y\cos(x^2+y^2)+xz\right)=x.$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)=\frac{\partial}{\partial z}xy=0.$$

Anm. Man kan göra processen kort genom att åberopa Sats 9 på s. 74.

Resultat: $\frac{\partial^2 f}{\partial x^2}=2\cos(x^2+y^2)-4x^2\sin(x^2+y^2)$,

$\frac{\partial^2 f}{\partial y^2}=2\cos(x^2+y^2)-4y^2\sin(x^2+y^2)$, $\frac{\partial^2 f}{\partial z^2}=0$,

$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-4xy\sin(x^2+y^2)+z$, $\frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial z\partial x}=y$

och $\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}=x$.

Övning 2.51 (s. 32)

$f(x,y)=g(x^2-y)=g(u)\ \wedge\ u=x^2-y.$

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}g(u)=g'(u)\frac{\partial u}{\partial x}=g'(u)\cdot 2x=2xg'(x^2-y);$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}g(u)=g'(u)\frac{\partial u}{\partial y}=g'(u)\cdot(-1)=-g'(x^2-y);$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}2xg'(u)=2g'(u)+2xg''(u)\frac{\partial u}{\partial x}=$$

$$=2g'(u)+4x^2g''(u).$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}(2xg'(u))=2xg''(u)\frac{\partial u}{\partial y}=-2xg''(u);$$

$$VL=2\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial x^2}+x\frac{\partial^2 f}{\partial y\partial x}=-2g'(u)+2g'(u)+4x^2g''(u)-$$

$$-2x^2g''(u)\equiv 2x^2g''(u)=0=HL\Leftrightarrow g''(u)=0\Leftrightarrow$$

$$\Leftrightarrow g(u)=C_1u+C_2\ \Rightarrow\ f(x,y)=C_1(x^2-y)+C_2.$$

Övning 2.52 (s. 32)

$$r=\sqrt{x^2+y^2}=(x^2+y^2)^{1/2}\Rightarrow\frac{\partial r}{\partial x}=\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x=\frac{x}{r}.$$

P.s.s. fås $\frac{\partial r}{\partial y}=\frac{y}{r}$ (eller p.g.a. symmetrin).

$u(x,y)=f(r)$;

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(r)=f'(r)\frac{\partial r}{\partial x}=xr^{-1}f'(r);\quad\frac{\partial u}{\partial y}=yr^{-1}f'(r).$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}(xr^{-1}f'(r))=r^{-1}f'(r)+x(-r^{-2})\frac{\partial r}{\partial x}f'(r)+$$

$$+xr^{-1}f''(r)\frac{\partial r}{\partial x}=r^{-1}f'(r)-x^2r^{-3}f'(r)+x^2r^{-2}f''(r)=$$

$$=\left(\frac{1}{r}-\frac{x^2}{r^3}\right)f'(r)+\frac{x^2}{r^2}f''(r).$$

P.g.a. symmetrin fås $\frac{\partial^2 u}{\partial y^2}=\left(\frac{1}{r}-\frac{y^2}{r^3}\right)f'(r)+\frac{y^2}{r^2}f''(r)$;

$$VL=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\left(\frac{2}{r}-\frac{x^2+y^2}{r^3}\right)f'(r)+\frac{x^2+y^2}{r^2}f''(r)=$$

$$=\left(\frac{2}{r}-\frac{r^2}{r^3}\right)f'(r)+\frac{r^2}{r^2}f''(r)=\frac{1}{r}f'(r)+f''(r)=r^2=HL\Leftrightarrow$$

$$\Leftrightarrow rf''(r)+f'(r)=r^3\Leftrightarrow(rf'(r))'=r^3\Leftrightarrow rf'(r)=\frac{1}{4}r^4+C_1$$

$$\Leftrightarrow f'(r)=\frac{1}{4}r^3+\frac{C_1}{r}\Leftrightarrow f(r)=\frac{1}{16}r^4+C_1\ln r+C_2.$$

Resultat: $u(x,y)=\frac{1}{16}(x^2+y^2)^2+A\ln(x^2+y^2)+B.$

Anm. Problemet har cylindersymmetri.

Övning 2.53 (s.33)

$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}-\frac{1}{r}\frac{\partial T}{\partial r}\ ;\ T(r,t)=f(u),\ u=\frac{r}{\sqrt{t}},\ r=\sqrt{x^2+y^2}$

$$\begin{cases}\frac{\partial T}{\partial r}=\frac{\partial}{\partial r}f(u)=f'(u)\frac{\partial u}{\partial r}=\frac{1}{\sqrt{t}}f'(u).\\ \frac{\partial^2 T}{\partial r^2}=\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}\right)=\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{t}}f'(u)\right)=\frac{1}{\sqrt{t}}\frac{\partial}{\partial r}f'(u)=\frac{1}{t}f''(u).\\ \frac{\partial T}{\partial t}=\frac{\partial}{\partial t}f(u)=f'(u)\frac{\partial u}{\partial t}=f'(u)\cdot\left(-\frac{1}{2}t^{-3/2}r\right)=-\frac{1}{2r^{3/2}}f'(u).\end{cases}$$

$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial r^2}-\frac{1}{r}\frac{\partial T}{\partial r}\Rightarrow -\frac{1}{2}t^{-3/2}r\,f'(u)=t^{-1}f''(u)-\frac{1}{r}\frac{1}{\sqrt{t}}f'(u)$

$\Leftrightarrow -\frac{1}{2}\frac{r}{\sqrt{t}}f'(u)=f''(u)-\frac{\sqrt{t}}{r}f'(u)\Leftrightarrow f''(u)=\left(\frac{1}{u}-\frac{u}{2}\right)f'(u)$

$\Leftrightarrow \frac{f''(u)}{f'(u)}=\frac{1}{u}-\frac{u}{2}\Leftrightarrow (\ln f'(u))'=\frac{1}{u}-\frac{u}{2}\Leftrightarrow \ln f'(u)=$

$=\ln u-\frac{1}{4}u^2+C_1\Leftrightarrow \ln f'(u)-\ln u=C_1-\frac{1}{4}u^2\Leftrightarrow$

$\Leftrightarrow \ln\frac{f'(u)}{u}=C_1-\frac{1}{4}u^2\Leftrightarrow \frac{f'(u)}{u}=\exp\{C_1-\frac{1}{4}u_2\}\Leftrightarrow$

$\Leftrightarrow f'(u)=C_2ue^{-u^2/4}\Leftrightarrow f(u)=-2C_2e^{-u^2/4}+C_3$

Resultat: $T(r,t)=f\left(\frac{r}{\sqrt{t}}\right)=Ae^{-r^2/4t}+B.$

Övning 2.54 (s.33)

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=u\ ;\ u(x,y,z)=f(r),\ r=\sqrt{x^2+y^2+z^2}.$

$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x=\frac{x}{r}\ ;$

P.g.a. (den sfäriska) symmetrin fås liknande uttryck för de andra koordinaterna y och z.

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(r)=f'(r)\frac{\partial r}{\partial x}=\frac{x}{r}f'(r).$

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}f'(r)\right)=\frac{\partial}{\partial x}(xr^{-1}f'(r))=r^{-1}f'(r)-$

$-xr^{-2}\frac{\partial r}{\partial x}f'(r)+xr^{-1}f''(r)\frac{\partial r}{\partial x}=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$

$=\frac{r^2-x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r).$

P.g.a. symmetrin fås liknande uttryck för y o z.

$\Delta u=u\Rightarrow \frac{3r^2-(x^2+y^2+z^2)}{r^3}f'(r)+\frac{x^2+y^2+z^2}{r^2}f''(r)=f(r)\Leftrightarrow$

$\Leftrightarrow \frac{3r^2-r^2}{r^3}f'(r)+\frac{r^2}{r^2}f''(r)=f(r)\Leftrightarrow \frac{2r^2}{r^3}f'(r)+f''(r)=f(r)$

$\Leftrightarrow f''(r)+\frac{2}{r}f'(r)=f(r)\ ;\ (*)$

$f(r)=\frac{1}{r}g(r)\Rightarrow f'(r)=\frac{1}{r}g'(r)-\frac{1}{r^2}g(r)\Rightarrow f''(r)=\frac{1}{r}g''(r)-$

$-\frac{2}{r^2}g'(r)+\frac{2}{r^3}g(r)\ \overset{(*)}{\Rightarrow}\ \frac{1}{r}g''(r)-\frac{2}{r^2}g'(r)+\frac{2}{r^3}g(r)+\frac{2}{r^2}g'(r)$

$-\frac{2}{r^3}g(r)=\frac{1}{r}g(r)\Leftrightarrow \frac{1}{r}g''(r)=\frac{1}{r}g(r)\Leftrightarrow g''(r)-g(r)=0$

$\Leftrightarrow g(r)=C_1e^r+C_2e^{-r}\Leftrightarrow f(r)=\frac{C_1e^r+C_2e^{-r}}{r}=\frac{C_1}{r}e^r+\frac{C_2}{r}e^{-r};$

$\lim_{r\to\infty}|f(r)|<\infty\Rightarrow C_1=0\Rightarrow f(r)=\frac{C_2}{r}e^{-r}.$

Resultat: $u(x,y,z)=\frac{C}{\sqrt{x^2+y^2+z^2}}e^{-\sqrt{x^2+y^2+z^2}}.$

Övning 2.55 (S. 33)

$\frac{\partial}{\partial x}e^f=e^f\frac{\partial f}{\partial x}\Rightarrow \frac{\partial^2}{\partial x^2}e^f=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}e^f\right)=\frac{\partial}{\partial x}\left(e^f\frac{\partial f}{\partial x}\right)=$

$=e^f\left(\frac{\partial f}{\partial x}\right)^2+e^f\frac{\partial^2 f}{\partial x^2}$ och liknande uttryck för x,z.

$$\Delta e^f = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)e^f = \frac{\partial^2}{\partial x^2}e^f+\frac{\partial^2}{\partial y^2}e^f+\frac{\partial^2}{\partial z^2}e^f =$$
$$= e^f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)+e^f\left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2\right)=$$
$$= e^f\Delta f + e^f|\mathrm{grad}\, f|^2 = e^f|\mathrm{grad}\, f|^2 = 0 \Leftrightarrow$$
$$\Leftrightarrow |\mathrm{grad}\, f|^2 = 0 \Leftrightarrow \mathrm{grad}\, f = \vec{0} \Leftrightarrow \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0\Leftrightarrow$$
$$\Leftrightarrow \underline{f(x,y,z)=C} \ \text{(konstant)}.$$

Övning 2.56 (s.33)

$$\underline{f(x,y)=\tilde f(u,v)\,;\quad u=x+y,\quad v=xy.}$$
$$\frac{\partial f}{\partial x} = \frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial \tilde f}{\partial u}+y\frac{\partial \tilde f}{\partial v}\,;$$
$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \tilde f}{\partial u}+y\frac{\partial \tilde f}{\partial v}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \tilde f}{\partial u}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial \tilde f}{\partial v}\right)$$
$$\begin{cases} \dfrac{\partial}{\partial y}\left(\dfrac{\partial \tilde f}{\partial u}\right) = \dfrac{\partial}{\partial u}\left(\dfrac{\partial \tilde f}{\partial u}\right)\dfrac{\partial u}{\partial y}+\dfrac{\partial}{\partial v}\left(\dfrac{\partial \tilde f}{\partial u}\right)\dfrac{\partial v}{\partial y} = \dfrac{\partial^2 \tilde f}{\partial u^2}+x\dfrac{\partial^2 \tilde f}{\partial v\partial u}\,; \\ \dfrac{\partial}{\partial y}\left(y\dfrac{\partial \tilde f}{\partial v}\right) = \dfrac{\partial \tilde f}{\partial v}+y\left(\dfrac{\partial}{\partial u}\left(\dfrac{\partial \tilde f}{\partial v}\right)\dfrac{\partial u}{\partial y}+\dfrac{\partial}{\partial v}\left(\dfrac{\partial \tilde f}{\partial v}\right)\dfrac{\partial v}{\partial y}\right) = \end{cases}$$
$$= \frac{\partial \tilde f}{\partial v}+y\left(\frac{\partial^2 \tilde f}{\partial u\partial v}+x\frac{\partial^2 \tilde f}{\partial v^2}\right) = \frac{\partial \tilde f}{\partial v}+y\frac{\partial^2 \tilde f}{\partial u\partial v}+xy\frac{\partial^2 \tilde f}{\partial v^2}\,;$$
$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 \tilde f}{\partial u^2}+(x+y)\frac{\partial^2 \tilde f}{\partial u\partial v}+xy\frac{\partial^2 \tilde f}{\partial v^2}+\frac{\partial \tilde f}{\partial v} = \underline{\frac{\partial^2 \tilde f}{\partial u^2}+u\frac{\partial^2 \tilde f}{\partial u\partial v}+}$$
$$\underline{+v\frac{\partial^2 \tilde f}{\partial v^2}+\frac{\partial \tilde f}{\partial v}}.$$

Övning 2.57 (s.33)

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}-6\frac{\partial^2 f}{\partial y^2}=1\,;\quad \underline{f(x,y)=\tilde f(u,v);\ u=x+\alpha y,\ v=x+\beta y}.$$
$$\frac{\partial f}{\partial x}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\,;\quad \frac{\partial f}{\partial y}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}=\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\,;$$

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$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}+\frac{\partial}{\partial v} \ \text{och}\ \frac{\partial}{\partial y}=\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}$$
$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\cdot\left(\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\right)=\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\right)+$$
$$+\frac{\partial}{\partial v}\left(\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\right)= \frac{\partial^2 \tilde f}{\partial u^2}+2\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{\partial^2 \tilde f}{\partial v^2}\,;$$
$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)=\frac{\partial}{\partial u}\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)+$$
$$+\frac{\partial}{\partial v}\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)=\alpha\frac{\partial^2 \tilde f}{\partial u^2}+(\alpha+\beta)\frac{\partial^2 \tilde f}{\partial u\partial v}+\beta\frac{\partial^2 \tilde f}{\partial v^2}\,;$$
$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\left(\alpha\frac{\partial}{\partial u}+\beta\frac{\partial}{\partial v}\right)\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)=\alpha\frac{\partial}{\partial u}\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)$$
$$+\beta\frac{\partial}{\partial v}\left(\alpha\frac{\partial \tilde f}{\partial u}+\beta\frac{\partial \tilde f}{\partial v}\right)=\alpha^2\frac{\partial^2 \tilde f}{\partial u^2}+2\alpha\beta\frac{\partial^2 \tilde f}{\partial u\partial v}+\beta^2\frac{\partial^2 \tilde f}{\partial v^2}.$$
$$VL=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}-6\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 \tilde f}{\partial u^2}+2\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{\partial^2 \tilde f}{\partial v^2}+\alpha\frac{\partial^2 \tilde f}{\partial u^2}+$$
$$+(\alpha+\beta)\frac{\partial^2 \tilde f}{\partial u\partial v}+\beta\frac{\partial^2 \tilde f}{\partial v^2}-6\left(\alpha^2\frac{\partial^2 \tilde f}{\partial u^2}+2\alpha\beta\frac{\partial^2 \tilde f}{\partial u\partial v}+\beta^2\frac{\partial^2 \tilde f}{\partial v^2}\right)=$$
$$=(1+\alpha-6\alpha^2)\frac{\partial^2 \tilde f}{\partial u^2}+(2+\alpha+\beta-12\alpha\beta)\frac{\partial^2 \tilde f}{\partial u\partial v}+(1+\beta-6\beta^2)\frac{\partial^2 \tilde f}{\partial v^2}$$
$$=1=HL.$$

$$\begin{cases}1+\alpha-6\alpha^2=0\\ 1+\beta-6\beta^2=0\end{cases}\Leftrightarrow\begin{cases}\alpha=\frac12\ \vee\ \alpha=-\frac13\\ \beta=\frac12\ \vee\ \beta=-\frac13\end{cases}\Rightarrow\begin{cases}\alpha=\frac12\\ \beta=-\frac13\end{cases}\quad (\text{t.ex.})$$
$$\frac{25}{6}\frac{\partial^2 \tilde f}{\partial u\partial v}=1\Leftrightarrow\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial v}\right)=\frac{6}{25}\Leftrightarrow\frac{\partial \tilde f}{\partial v}=\frac{6}{25}u+\phi(v)\Leftrightarrow$$
$$\Leftrightarrow \tilde f(u,v)=\frac{6}{25}uv+F(u)+G(v)\quad \left(G(v)=\int\phi(v)\,dv\right)$$
$$\Leftrightarrow f(x,y)=\frac{6}{25}\left(x+\tfrac12 y\right)\left(x-\tfrac13 y\right)+F\left(x+\tfrac12 y\right)+G\left(x-\tfrac13 y\right)\Leftrightarrow$$
$$\Leftrightarrow f(x,y)=\frac{1}{25}(2x+y)(3x-y)+F\left(x+\tfrac12 y\right)+G\left(x-\tfrac13 y\right).$$

Övning 2.58 (s. 34)

$$x\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial x}=xe^{-2y}\ ;\ f(x,y)=\tilde f(u,v)\ ;\ \begin{cases}u=xe^{-y}\\ v=y\end{cases}.$$

a) $$\frac{\partial f}{\partial x}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}=e^{-y}\frac{\partial \tilde f}{\partial u}\ ;$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^{-y}\frac{\partial \tilde f}{\partial u}\right)=e^{-y}\frac{\partial}{\partial x}\left(\frac{\partial \tilde f}{\partial u}\right)=e^{-2v}\frac{\partial^2 \tilde f}{\partial u^2}\ ;$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(e^{-y}\frac{\partial \tilde f}{\partial u}\right)=-e^{-y}\frac{\partial \tilde f}{\partial u}+$$

$$+e^{-v}\frac{\partial}{\partial y}\left(\frac{\partial \tilde f}{\partial u}\right)=-e^{-v}\frac{\partial \tilde f}{\partial u}+e^{-v}\left(\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial \tilde f}{\partial u}\right)\frac{\partial v}{\partial y}\right)$$

$$=-e^{-v}\frac{\partial \tilde f}{\partial u}+e^{-v}\left(-xe^{-y}\frac{\partial^2 \tilde f}{\partial u^2}+\frac{\partial^2 \tilde f}{\partial u\partial v}\right)=$$

$$=-e^{-v}\frac{\partial \tilde f}{\partial u}+e^{-v}\left(-u\frac{\partial^2 \tilde f}{\partial u^2}+\frac{\partial^2 \tilde f}{\partial u\partial v}\right)=$$

$$=-e^{-v}\frac{\partial \tilde f}{\partial u}-ue^{-v}\frac{\partial^2 \tilde f}{\partial u^2}+e^{-v}\frac{\partial^2 \tilde f}{\partial u\partial v}\ ;$$

$$VL=x\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial x}=xe^{-2y}\frac{\partial^2 \tilde f}{\partial u^2}+e^{-v}\frac{\partial^2 \tilde f}{\partial u\partial v}-e^{-v}\frac{\partial \tilde f}{\partial u}-$$

$$-ue^{-v}\frac{\partial^2 \tilde f}{\partial u^2}+e^{-v}\frac{\partial \tilde f}{\partial u}=ue^{-v}\frac{\partial^2 \tilde f}{\partial u^2}+e^{-v}\frac{\partial^2 \tilde f}{\partial u\partial v}-e^{-v}\frac{\partial \tilde f}{\partial u}-$$

$$-ue^{-v}\frac{\partial^2 \tilde f}{\partial u^2}+e^{-v}\frac{\partial \tilde f}{\partial u}=e^{-v}\frac{\partial^2 \tilde f}{\partial u\partial v}=xe^{-2y}=ue^{-v}\Leftrightarrow$$

$$\Leftrightarrow \frac{\partial^2 \tilde f}{\partial u\partial v}=u.$$

b) $$\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial v}\right)=u\Leftrightarrow \frac{\partial \tilde f}{\partial v}=\frac{1}{2}u^2+\phi(v)\Leftrightarrow \tilde f(u,v)=\frac{1}{2}u^2v+$$

$$+F(v)+G(u)\Leftrightarrow \underline{f(x,y)=\frac{1}{2}x^2ye^{-2y}+F(y)+G(xe^{-y})}.$$

Övning 2.59 (s. 34)

$$x\frac{\partial^2 f}{\partial x^2}-y\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial x}=0\ ;\ f(x,y)=\tilde f(u,v)\ ;\ u=y,\ v=xy.$$

$$\frac{\partial f}{\partial x}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial \tilde f}{\partial v}\ ;$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial \tilde f}{\partial v}\right)=y\frac{\partial}{\partial v}\left(y\frac{\partial \tilde f}{\partial v}\right)=u^2\frac{\partial^2 \tilde f}{\partial v^2}\ ;$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(y\frac{\partial \tilde f}{\partial v}\right)=\frac{\partial \tilde f}{\partial v}+y\frac{\partial}{\partial y}\left(\frac{\partial \tilde f}{\partial v}\right)=$$

$$=\frac{\partial \tilde f}{\partial v}+y\left(\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial \tilde f}{\partial v}\right)\frac{\partial v}{\partial y}\right)=$$

$$=\frac{\partial \tilde f}{\partial v}+y\left(\frac{\partial^2 \tilde f}{\partial u\partial v}+x\frac{\partial^2 \tilde f}{\partial v^2}\right)=\frac{\partial \tilde f}{\partial v}+u\frac{\partial^2 \tilde f}{\partial u\partial v}+v\frac{\partial^2 \tilde f}{\partial v^2}\ ;$$

$$VL=x\frac{\partial^2 f}{\partial x^2}-y\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial x}=\frac{v}{u}\left(u^2\frac{\partial^2 \tilde f}{\partial v^2}\right)-u\left(\frac{\partial \tilde f}{\partial v}+u\frac{\partial^2 \tilde f}{\partial u\partial v}\right.$$

$$\left.+v\frac{\partial^2 \tilde f}{\partial v^2}\right)+u\frac{\partial \tilde f}{\partial v}=uv\frac{\partial^2 \tilde f}{\partial v^2}-u\frac{\partial \tilde f}{\partial v}-u^2\frac{\partial^2 \tilde f}{\partial u\partial v}-$$

$$-uv\frac{\partial^2 \tilde f}{\partial v^2}+u\frac{\partial \tilde f}{\partial v}=-u^2\frac{\partial^2 \tilde f}{\partial u\partial v}=0=HL\Leftrightarrow \frac{\partial^2 \tilde f}{\partial u\partial v}=0$$

$$\Leftrightarrow \frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial v}\right)=0\Leftrightarrow \frac{\partial \tilde f}{\partial v}=\phi(v)\Leftrightarrow \tilde f(u,v)=F(u)+G(v)\Leftrightarrow$$

$$\Leftrightarrow \underline{f(x,y)=F(y)+G(xy)}.$$

Lokala undersökningar

Övning 2.60 (s. 34)

a) $\underline{f(\mathbf{x})=(1+x_1+2x_2)^2}$

$$u=x_1-1\ \wedge\ v=x_2-1\Leftrightarrow x_1=1+u\ \wedge\ x_2=1+v\ ;$$

$$f(x_1,x_2)=f(1+u,1+v)=(1+1+u+2+2v)^2=$$

$$=(4+u+2v)^2=\text{(utvecklas)}=$$

$$=16+u^2+4v^2+8u+16v+4uv=$$

$= 16 + 8(x_1-1) + 16(x_2-1) + (x_1-1)^2 + 4(x_2-1)^2 + 4(x_1-1)(x_2-1).$

Resultat: $P_1(x_1,x_2) = 16 + 8(x_1-1) + 16(x_2-1).$

$P_2(x_1,x_2) = P_1(x_1,x_2) + (x_1-1)^2 + 4(x_1-1)(x_2-1) + 4(x_2-1)^2.$

b) $f(x_1,x_2) = (1+x_1+2x_2)^{-1}$;

$f(1+u,1+v) = (4+u+2v)^{-1} = \left(4\left(1+\frac{u+2v}{4}\right)\right)^{-1} =$

$= 4^{-1}\left(1+\frac{u+2v}{4}\right)^{-1} =$ (binomialsatsen) $=$

$= \frac{1}{4}\left(1 - \frac{u+2v}{4} + \left(\frac{u+2v}{4}\right)^2 + \text{högregradstermer}\right) =$

$= \frac{1}{4} + \frac{1}{16}(u+2v) + \frac{1}{64}(u+2v)^2 + \text{högregradstermer} =$

$= \frac{1}{4} + \frac{1}{16}u + \frac{1}{8}v + \frac{1}{64}u^2 + \frac{1}{16}uv + \frac{1}{16}v^2 + \text{annat} =$

$= \frac{1}{4} + \frac{1}{16}(x_1-1) + \frac{1}{8}(x_2-1) + \frac{1}{64}(x_1-1)^2 + \frac{1}{16}(x_1-1)(x_2-1) +$

$+ \frac{1}{16}(x_2-1)^2 +$ (allt annat som inte behövs).

Resultat: $P_1(x_1,x_2) = \frac{1}{4} + \frac{1}{16}(x_1-1) + \frac{1}{8}(x_2-1)$;

$P_2(x_1,x_2) = \frac{1}{64}(x_1-1)^2 + \frac{1}{16}(x_1-1)(x_2-1) + \frac{1}{16}(x_2-1)^2 + P_1(x_1,x_2).$

Anm. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots + (-1)^{n+1}\cdot\frac{x^{n+1}}{1+x}.$

I facit går man den "normala" vägen.

Lägg märke till att Taylorpolynomet för elementära funktioner är entydigt (unikt). Detta visas inte i grundboken, tyvärr.

Övning 2.61 (S.34)

Anm. $(1+u)^{1/2} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + O(u^3).$

a) $f(x,y) = (1+x+y^2)^{1/2} = (1+(x+y^2))^{1/2} = (u = x+y^2) =$

$= 1 + \frac{1}{2}(x+y^2) - \frac{1}{8}(x+y^2)^2 +$ (högre termer)

$= 1 + \frac{1}{2}x + \frac{1}{2}y^2 - \frac{1}{8}x^2 - \frac{1}{8}y^4 - \frac{1}{4}xy^2 + \text{annat} =$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{2}y^2 +$ (högre termer).

Resultat: $P_2(x,y) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{2}y^2.$

b) $f(x,y) = (x+1)^{y+1} = e^{(y+1)\ln(1+x)} =$

$= e^{(y+1)(x - x^2/2 + O(x^3))} =$

$= e^{x - x^2/2 + xy + \text{högre}} =$

$= 1 + (x - \frac{1}{2}x^2 + xy) + \frac{1}{2}(x - \frac{1}{2}x^2 + xy)^2 + \cdots =$

$= 1 + x - \frac{1}{2}x^2 + xy + \frac{1}{2}x^2 + \cdots = 1 + x + xy + \cdots$

Resultat: $P_2(x,y) = 1 + x + xy.$

Anm. Jag har utnyttjat utvecklingarna $\ln(1+u) = u - \frac{1}{2}u^2 + O(u^3)$, $e^v = 1 + v + \frac{1}{2}v^2 + O(v^3).$

Övning 2.62 (S.34)

a) $f(a+h, b+k) = f(a,b) + f'_x(a,b)h + f'_y(a,b)k +$

$$+\frac{1}{2!}\left(f''_{xx}(a,b)h^2+2f''_{xy}(a,b)hk+f''_{yy}(a,b)k^2\right)+O(|t|^3)$$

där $|t|=\sqrt{h^2+k^2}$.

En annan variant är

$$f(x,y)=f(a,b)+f'_x(a,b)(x-a)+f'_y(a,b)(y-b)+$$
$$+\frac{1}{2}\left(f''_{xx}(a,b)(x-a)^2+2f''_{xy}(a,b)(x-a)(y-b)+\right.$$
$$\left.+f''_{yy}(a,b)(y-b)^2\right)+O\left(\left(\sqrt{(x-a)^2+(y-b)^2}\right)^3\right).$$

Anm. (h,k) kallas lokala koordinater.

b) $f(x,y)=(1+x+y)^{1/2}$, $P=(1,0)$.

$$f(x,y)=f(1+u,v)=(2+u+v)^{1/2}=\sqrt{2}\left(1+\frac{u+v}{2}\right)^{1/2}=$$
$$=\sqrt{2}\left(1+\frac{1}{2}\frac{u+v}{2}-\frac{1}{8}\left(\frac{u+v}{2}\right)^2+\dots\right)=$$
$$=\sqrt{2}\left(1+\frac{u+v}{4}-\frac{u^2+2uv+v^2}{32}+\dots\right)=$$
$$=\sqrt{2}\left(1+\frac{1}{4}(x-1)+\frac{1}{4}y-\frac{1}{32}(x-1)^2-\frac{1}{16}(x-1)y-\frac{1}{32}y^2+\dots\right)$$

Resultat: $f(x,y)=\sqrt{2}+\frac{\sqrt{2}}{4}(x-1)+\frac{\sqrt{2}}{4}y-\frac{\sqrt{2}}{32}(x-1)^2-$
$-\frac{\sqrt{2}}{16}(x-1)y-\frac{\sqrt{2}}{32}y^2+O\left(\left(\sqrt{(x-1)^2+y^2}\right)^3\right).$

Övning 2.63 (s.34)

a) $Q(h,k)=h^2+6k^2+4hk=h^2+4hk+4k^2+2k^2=$

$$=\underbrace{(h+2k)^2+2k^2>0}_{(h,k)\neq(0,0)}\ \wedge\ Q(0,0)=0.$$

Q är positiv definit.

b) $Q(h,k,l)=h^2+2k^2+8l^2+2hk+2hl=$

$$=(h^2+k^2+l^2+2hk+2hl+2kl)+(k^2-2kl+7l^2)=$$
$$=(h+k+l)^2+(k^2-2kl+l^2)+6l^2=$$
$$=(h+k+l)^2+(k-l)^2+6l^2;$$

$$\begin{cases} h+k+l=0\\ k-l=0\\ l=0\end{cases}\Leftrightarrow h=k=l=0\Rightarrow Q \text{ positiv definit.}$$

Övning 2.64 (s.35)

a) $Q(h,k)=h^2+k^2>0$ för $(h,k)\neq(0,0)$.

Q är positiv definit.

b) $Q(h,k)=h^2-k^2\Rightarrow\begin{cases}Q(1,0)>0\\ Q(0,1)<0\end{cases}\Rightarrow Q$ indefinit.

c) $Q(h,k)=hk\Rightarrow\begin{cases}Q(1,1)>0\\ Q(1,-1)<0\end{cases}\Rightarrow Q$ indefinit.

d) $\begin{cases}Q(h,k)=h^2+k^2+hk\equiv\left(h+\frac{k}{2}\right)^2+\frac{3}{4}k^2\\ h+\frac{k}{2}=0=k\Leftrightarrow h=k=0\end{cases}\Rightarrow Q$ pos. definit.

e) $Q(h,k)=(h+k)^2\geq 0\ \wedge\ Q(1,-1)=0\Rightarrow Q$ pos. semidefinit.

f) $Q(h,k)=(h+2k)^2-3k^2 \Rightarrow Q$ indefinit.

Övning 2.65 (s. 35)

a) $Q(h_1,h_2,h_3)=h_1^2+h_2^2+h_3^2>0$ och $Q(0,0,0)=0 \Rightarrow$

$\Rightarrow Q$ pos. definit.

b) $Q(h_1,h_2,h_3)=h_1^2+h_2^2-h_3^2 \Rightarrow \begin{cases} Q(1,1,1)>0 \\ Q(1,1,2)<0 \end{cases} \Rightarrow$

$\Rightarrow Q$ indefinit.

c) $Q(h_1,h_2,h_3)=h_1^2+h_2^2 \geq 0 \wedge Q(0,0,1)=0 \Rightarrow$

$\Rightarrow Q$ pos. semidefinit.

d) $Q(h_1,h_2,h_3)=h_1h_3 \Rightarrow Q(1,1,1)>0 \wedge Q(1,1,-1)<0 \Rightarrow$

$\Rightarrow Q$ indefinit.

e) $Q(h_1,h_2,h_3)=h_1^2-h_2^2-h_3^2+2h_1h_2+4h_2h_3$;

$Q(1,0,0)>0 \wedge Q(0,0,1)<0 \Rightarrow Q$ indefinit.

f) $Q(h_1,h_2,h_3)=h_1^2+2h_2^2+2h_3^2+2h_1h_2-2h_1h_3+2h_2h_3$;

$Q(1,1,1)=6>0 \wedge Q(0,1,-\frac{1}{2})=-\frac{3}{4}<0 \Rightarrow Q$ indefinit.

Övning 2.66 (s. 35)

$f(x,y)=3x^2+3xy+y^2+y^3$

$\frac{\partial f}{\partial x}=6x+3y$, $\frac{\partial f}{\partial y}=3x+2y+3y^2$; forts.

$\frac{\partial^2 f}{\partial x^2}=6$, $\frac{\partial^2 f}{\partial y^2}=2+6y$, $\frac{\partial^2 f}{\partial x\partial y}=3$.

Kritiska punkter

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y} \Rightarrow \begin{cases} 6x+3y=0 \\ 3x+2y+3y^2=0 \end{cases} \Leftrightarrow \begin{cases} y=-2x \\ 3x-4x+12x^2=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x(12x-1)=0 \\ y=-2x \end{cases} \Leftrightarrow \begin{cases} x=0 \vee x=\frac{1}{12} \\ y=-2x \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=\frac{1}{12} \\ y=-\frac{1}{6} \end{cases};$$

Extrempunkter

$(x,y)=(0,0) \Rightarrow Q(h,k)=f''_{xx}(0,0)h^2+2f''_{xy}(0,0)hk+$
$+f''_{yy}(0,0)k^2=6h^2+6hk+2k^2=6(h^2+hk+\frac{1}{3}k^2)=$
$=6(h^2+hk+\frac{1}{4}k^2-\frac{1}{4}k^2+\frac{1}{3}k^2)=6(h+\frac{1}{2}k)^2+\frac{1}{2}k^2$;

Q är positiv definit, så $(0,0)$ är lokal minimipunkt; $f(0,0)=0$.

$(x,y)=(\frac{1}{12},-\frac{1}{6}) \Rightarrow Q(h,k)=6h^2+6hk+k^2=6(h-\frac{1}{2}k)^2$
$-\frac{1}{2}k^2$; Q är indefinit, så $(\frac{1}{12},-\frac{1}{6})$ är ingen extrempunkt.

Resultat: Den enda lokala extrempunkten är $(0,0)$; den är en lokal minimipunkt.

Anm. Ta en titt på författarnas facit också.

Övning 2.67 (s.35)

$f(x,y)=x^3y^2+27xy+27y$.

$\frac{\partial f}{\partial x}=3x^2y^2+27y$, $\frac{\partial f}{\partial y}=2x^3y+27x+27$;

$\frac{\partial^2 f}{\partial x^2}=6xy^2$, $\frac{\partial^2 f}{\partial y^2}=2x^3$, $\frac{\partial^2 f}{\partial x\partial y}=6x^2y+27$.

Kritiska punkter

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}\Rightarrow\begin{cases}3y(yx^2+9)=0\\2x^3y+27x+27=0\end{cases}\Leftrightarrow\begin{cases}y=0\vee y=-\frac{9}{x^2}\\2x^3y+27x+27=0\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}y=0\\27x+27=0\end{cases}\vee\begin{cases}y=-9/x^2\\9x+27=0\end{cases}\Leftrightarrow\begin{cases}x=-1\\y=0\end{cases}\vee\begin{cases}x=-3\\y=-1\end{cases};$$

$\underline{(x,y)=(-1,0)}\Rightarrow Q(h,k)=f''_{xx}(-1,0)h^2+2f''_{xy}(-1,0)hk+$
$+f''_{yy}(-1,0)k^2=54hk-2k^2\Rightarrow Q(1,1)>0\wedge Q(0,1)<0$
$\Rightarrow Q$ indefinit $\Rightarrow(-1,0)$ ingen extrempunkt.

$\underline{(x,y)=(-3,-1)}\Rightarrow Q(h,k)=-18h^2-54hk-54k^2=$
$=-18(h^2+3hk+3k^2)=-18((h+\frac{3}{2}k)^2+\frac{3}{4}k^2)\Rightarrow$
$\Rightarrow$ negativ definit $\Rightarrow(-3,-1)$ maximipunkt.

Resultat: Den enda extrempunkten är $(-3,-1)$; denna är en lokal maximipunkt.

Anm. Punkten $(-1,0)$ är en s.k. sadelpunkt.

Övning 2.68 (s.35)

a) $f(x,y)=(1+\sin(x+y))\ln(1+2x+y)-2x-y=$
$=(1+x+y+O(|\mathbf{x}|^3))(2x+y-\frac{1}{2}(2x+y)^2+O(|\mathbf{x}|^3))$
$-2x-y=2x+y-\frac{1}{2}(2x+y)^2+(x+y)(2x+y)+$
$+O(|\mathbf{x}|^3)-2x-y=2xy+y^2+O(|\mathbf{x}|^3)\Rightarrow$
$\Rightarrow Q(x,y)=2xy+y^2\Rightarrow Q$ indefinit $\Rightarrow$ $\underline{(0,0)\text{ ingen extrempunkt}}$.

b) $f(x,y)=4x^2+12xy+9y^2+x^4=(2x+3y)^2+x^4\geq 0\Rightarrow$
$\Rightarrow(0,0)$ minimipunkt.

Anm. I detta fall har jag utnyttjat definitionen direkt: $\Delta f=f(x,y)-f(0,0)\geq 0$. Se definition 7 på s. 86 i grundboken.

c) $f(x,y,z)=e^{xyz}(1-\arctan(x^2+y^2+2z^2))=$
$=(1+xyz+O(|\mathbf{x}|^6))(1-x^2-y^2-2z^2+O(|\mathbf{x}|^6))$
$=1-x^2-y^2-2z^2+O(|\mathbf{x}|^3)\Rightarrow\Delta f=f(x,y,z)-$
$-f(0,0,0)=f(\mathbf{x})-1=-(x^2+y^2+2z^2)+O(|\mathbf{x}|^3)$;

För små $|\mathbf{x}|$ har vi $\Delta f\approx-(x^2+y^2+2z^2)<0\Rightarrow$
$\Rightarrow(0,0,0)$ maximipunkt.

d) $f(\mathbf{x}) = 1+x^2+2y^2+4z^2-2xy+6yz-2xz =$

$= 1+x^2+y^2+z^2-2xy+2yz-2xz+y^2+3z^2+4yz =$

$= 1+(x-y-z)^2+(y+2z)^2-z^2 \Leftrightarrow \Delta f = f(\mathbf{x})-1 =$

$= f(\mathbf{x})-f(\mathbf{0}) = (x-y-z)^2+(y+2z)-z^2$ indefinit $\Rightarrow$

$\Rightarrow (0,0,0)$ ingen extrempunkt.

Övning 2.69 (s. 35)

a) $f: \mathbb{R}^n \to \mathbb{R},\ f \in C^1,\ \mathbf{a} \in D_f$.

(i) Stationär (kritisk) punkt.

$\mathbf{a}$ stationär $\Leftrightarrow \operatorname{grad} f(\mathbf{a}) = \mathbf{0} \Leftrightarrow f_i'(\mathbf{a}) = 0$ för $i = 1, 2, 3, \ldots, n$.

(ii) Lokalt maximum

f uppvisar lokalt maximum i $\mathbf{a}$ om det finns en öppen omgivning

$$O_\varepsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}-\mathbf{a}| < \varepsilon\}$$

sådan att $\forall \mathbf{x} \in O_\varepsilon(\mathbf{a}) \cap D_f : f(\mathbf{a}) \geq f(\mathbf{x})$.

(iii) Positiv definit kvadratisk form

En kvadratisk form $Q(\mathbf{h})$, $\mathbf{h} = (h_1, h_2, \ldots, h_n)$, säges vara positiv definit om $Q(\mathbf{h}) > 0$ för $\mathbf{h} \neq \mathbf{0}$ och $Q(\mathbf{0}) = 0$.

b) $f: \mathbb{R}^n \to \mathbb{R},\ f \in C^k\ (k \geq 3)$.

$\mathbf{a} \in D_f$ är en stationärpunkt till f, dvs. en punkt s.a. $\operatorname{grad} f(\mathbf{a}) = \mathbf{0}$. $O_\varepsilon(\mathbf{a})$ är en öppen omgivning till $\mathbf{a}$. Om $\mathbf{a}+\mathbf{h} \in O_\varepsilon(\mathbf{a}) \cap D_f$ så kan vi utveckla f i en Taylorserie enligt följande:

$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \frac{1}{2}\mathbf{h} \cdot \mathbb{H}(\mathbf{a})\mathbf{h}^T + O(|\mathbf{h}|^3)$.

$\mathbb{H}(\mathbf{a})$ är Hessianen eller Hessenmatrisen

$$\mathbb{H}(\mathbf{a}) = \begin{bmatrix} f''_{xx}(\mathbf{a}) & f''_{xy}(\mathbf{a}) \\ f''_{yx}(\mathbf{a}) & f''_{yy}(\mathbf{a}) \end{bmatrix};$$

Om den kvadratiska formen $Q(\mathbf{h}) = \mathbf{h} \cdot \mathbb{H} \cdot \mathbf{h}^T$ är positiv definit uppvisar f lokalt minimum i $\mathbf{a}$; om Q är negativ definit är $\mathbf{a}$ en (lokal) maximipunkt; om Q är semidefinit kan inga slutsatser dras; om den slutligen är indefinit så är $\mathbf{a}$ en s.k. sadelpunkt, dvs. ingen extrempunkt.

c) $f(x,y)=x^2(1+y)+y(3x+y)-2=-2+x^2+x^2y+3xy+y^2$

$=-2+x^2+3xy+y^2+x^2y=-2+x^2+3xy+y^2+$

$+O(|\mathbf{x}|^3)=-2+(x+\frac{3}{2}y)^2-\frac{1}{4}y^2+O(|\mathbf{x}|^3)$;

$\Leftrightarrow f(\mathbf{x})+2=\underbrace{(x+\frac{3}{2}y)^2-\frac{1}{4}y^2}_{\text{indefinit}}+O(|\mathbf{x}|^3)$.

Resultat: Origo är ingen extrempunkt till f.

Övning 2.70 (s. 35)

$f(x,y,z)=(x+xy+yz)e^x$

$\frac{\partial f}{\partial x}=(1+x+y+xy+yz)e^x,\ \frac{\partial f}{\partial y}=(x+z)e^x,\ \frac{\partial f}{\partial z}=ye^x$

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0\Rightarrow\begin{cases}1+x+y+xy+yz=0\\x+z=0\\y=0\end{cases}\Leftrightarrow\begin{cases}x=-1\\y=0\\z=1\end{cases}$;

$\frac{\partial^2 f}{\partial x^2}=(1+x+2y+xy+yz)e^x,\ \frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial z^2}=0,\ \frac{\partial^2 f}{\partial x\partial z}=ye^x,$

$\frac{\partial^2 f}{\partial x\partial y}=(1+x+z)e^x,\ \frac{\partial^2 f}{\partial y\partial z}=e^x$

$(x,y,z)=(-1,0,1)\Rightarrow f''_{xx}=f''_{xz}=f''_{yy}=f''_{zz}=0\wedge f''_{xy}=f''_{yz}=\frac{1}{e}$

$\Rightarrow 2Q(h,k,l)=\frac{2}{e}(hk+kl)$ indefinit $\Rightarrow (-1,0,1)$

ingen extrempunkt.

Resultat: $(-1,0,1)$ är den enda stationära punkten till f; den är ingen extrempunkt dock.

Differentialer

Övning 2.71 (s. 35)

a) $f(x,y)=\sin(xy^2)$

$df=d(\sin(xy^2))=\cos(xy^2)\,d(xy^2)=\cos(xy^2)\cdot$

$\cdot(dx\cdot y^2+x\,dy^2)=(y^2dx+2xy\,dy)\cos(xy^2)$.

Annan metod

$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=y^2\cos(xy^2)dx+2xy\cos(xy^2)dy$

$=(y^2dx+2xy\,dy)\cdot\cos(xy^2)$.

b) $f(x,y)=\ln(x+2y)$

$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=\frac{1}{x+2y}dx+\frac{2}{x+2y}dy=\frac{dx+2dy}{x+2y}$.

c) $f(x,y,z)=\sin(xyz)$.

$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=\cos(xyz)\,yz\,dx+$

$+\cos(xyz)\cdot xz\,dy+\cos(xyz)\,xy\,dz=$

$=\cos(xyz)(yz\,dx+xz\,dy+xy\,dz)$.

d) $f(p,V,T)=\frac{pV}{T}$.

$df=\frac{\partial f}{\partial p}dp+\frac{\partial f}{\partial V}dV+\frac{\partial f}{\partial T}dT=\frac{V}{T}dp+\frac{p}{T}dV-\frac{pV}{T^2}dT$.

Anm. $\ln f=\ln p+\ln V-\ln T\Rightarrow\frac{df}{f}=\frac{dp}{p}+\frac{dV}{V}-\frac{dT}{T}$

$\Leftrightarrow df=f\cdot(\frac{dp}{p}+\frac{dV}{V}-\frac{dT}{T})=\frac{V}{T}dp+\frac{p}{T}dV-\frac{pV}{T^2}dT$.

Övning 2.72 (s.36)

$f(x,y)=\frac{y^2}{x} \Rightarrow f(2+\Delta x, 1+\Delta y)-f(2,1)=\frac{(1+\Delta y)^2}{2+\Delta x}-\frac{1}{2};$

$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=-\frac{y^2}{x^2}dx+2\frac{y}{x}dy;$

a) $\begin{cases} \Delta f=\frac{1{,}3^2}{2{,}1}-\frac{1}{2}=0{,}305 \\ df=-\frac{1}{4}\cdot 0{,}1+0{,}3=0{,}275 \end{cases} \Rightarrow \Delta f-df=\underline{0{,}030}.$

b) $\begin{cases} \Delta f=\frac{1{,}03^2}{2{,}01}-\frac{1}{2}=0{,}0278 \\ df=-\frac{1}{4}\cdot 0{,}01+0{,}03=0{,}0275 \end{cases} \Rightarrow \Delta f-df=\underline{0{,}0003}.$

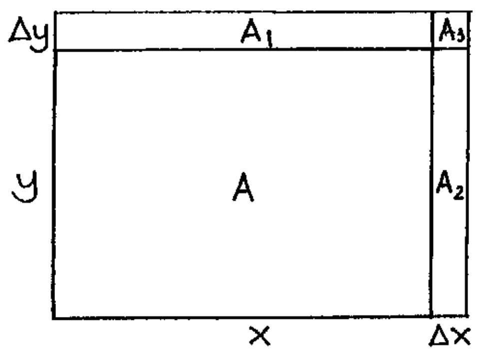
Övning 2.73 (s.36)

$f(x,y)=xy \Rightarrow \Delta f=f(x+\Delta x, y+\Delta y)-f(x,y)=$

$=(x+\Delta x)\cdot(y+\Delta y)-xy=\underline{x\Delta y+y\Delta x+\Delta x\cdot\Delta y};\ (*)$

$df=d(xy)=ydx+xdy=y\Delta x+x\Delta y;\ (**)$

$(*)-(**)=\Delta f-df=\Delta x\cdot\Delta y.$



forts.

$\underline{f=A,\ \Delta f=A_1+A_2+A_3;\ df=A_1+A_2;\ \Delta f-df=A_3.}$

Övning 2.74 (s.36)

$f(x,y)=(3+x+\sqrt{1-y})^{-1/2} \Rightarrow \begin{cases} \frac{\partial f}{\partial x}=-\frac{1}{2}(3+x+\sqrt{1-y})^{-3/2} \\ \frac{\partial f}{\partial y}=-\frac{1}{2}(3+x+\sqrt{1-y})^{-3/2}\left(\frac{-1/2}{\sqrt{1-y}}\right). \end{cases}$

$df=f'_x(0,0)dx+f'_y(0,0)dy=f'_x(0,0)x+f'_y(0,0)y=$

$=-\frac{1}{2}\cdot\frac{1}{4\cdot 2}x+\frac{1}{4}\frac{1}{4\cdot 2}y=-\frac{x}{16}+\frac{y}{32} \Rightarrow f(x,y)\approx f(0,0)+$

$+df(0,0)=\frac{1}{2}-\frac{x}{16}+\frac{y}{32}.$

Blandade problem

Övning 2.75 (S.36)

a) f differentierbar i $x=a \Leftrightarrow$ det existerar konstanter A_1 och A_2 s.a.

$$\Delta f=f(a+h)-f(a)=A_1h+A_2k+O(|h|^2).$$

Vi har bl.a. $A_1=f'_x(a)$ och $A_2=f'_y(a)$.

Men $\operatorname{grad} f(a)=(A_1,A_2)=(f'_x(a),f'_y(a))$.

Påståendet är sant.

b) $\operatorname{grad} f(x,y)$ är en normalvektor till ytan $z=f(x,y)$ i punkten $(x,y,f(x,y))$ och inte i punkten $(a,b,f(a,b))$; påståendet är <u>falskt</u>.

c) $\operatorname{grad} f(a,b)$ är normalvektor till ytan $z=f(x,y)$ i punkten $(a,b,f(a,b))$ och inte en tangentvektor i samma punkt; påståendet är <u>falskt</u>.

d) α extrempunkt $\Rightarrow f'_x(\alpha)=f'_y(\alpha)=0$, s.a. även $\operatorname{grad} f(\alpha)=(f'_x(\alpha),f'_y(\alpha))=(0,0)=\mathbf{0}$. <u>Sant</u>, alltså.

e) $f(x,y)=C \Rightarrow \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \operatorname{grad} f(x,y)=\mathbf{0}$. <u>Sant</u>.

f) $\operatorname{grad} f(a,b)$ är normalvektor till nivåkurvan $f(x,y)=f(a,b)$ i punkten $(a,b,f(a,b))$. Om (x,y) är en löpande punkt på tangenten, så är $\operatorname{grad} f(a,b)(x-a,y-b)=\frac{\partial f}{\partial x}(a,b)(x-a)+\frac{\partial f}{\partial y}(a,b)(y-b)=$ $=0$, så påståendet är <u>sant</u>.

g) $\operatorname{grad} f(x,y)=0 \Rightarrow \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow f(x,y)=\phi(x)=\psi(y)$ $\Rightarrow \phi(x)=\psi(y)=C=f(x,y)$; påståendet är <u>sant</u>.

h) Enligt Sats 7 på s. 67 är detta <u>sant</u>.

<u>Anm</u>. <u>Påstående</u> är genus neutrum. I facit svarar författarna i genus utrum (reale). Man menar kanske <u>utsaga</u> i stället. Eller?

<u>Övning 2.76 (s. 37)</u>

$u(r)=u(x,y,z)=f(r)$, $r=\sqrt{x^2+y^2+z^2}$;

$\frac{\partial r}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)^{1/2}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x=\frac{x}{r}$;

P.g.a. symmetrin fås analogt $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(r)=f'(r)\frac{\partial r}{\partial x}=\frac{x}{r}f'(r)=x\cdot r^{-1}f'(r)$.

$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}(xr^{-1}f'(r))=\frac{1}{r}f'(r)-\frac{x^2}{r^3}f'(r)+\frac{x^2}{r^2}f''(r)$

Symmetrin ger liknande uttryck för y och z.

$$VL=\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\left(\frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}\right)f'(r)+$$
$$+\frac{x^2+y^2+z^2}{r^2}f''(r)=\left(\frac{3}{r}-\frac{r^2}{r^3}\right)f'(r)+\frac{r^2}{r^2}f''(r)=$$
$$=\frac{2}{r}f'(r)+f''(r)=1=HL \Leftrightarrow r\cdot f''(r)+2f'(r)=r$$
$$\Leftrightarrow r^2f''(r)+2rf'(r)=r^2 \Leftrightarrow (r^2f'(r))'=r^2 \Leftrightarrow r^2f'(r)=$$
$$=\frac{1}{3}r^3+C_1 \Leftrightarrow f'(r)=\frac{1}{3}r+\frac{C_1}{r^2} \Leftrightarrow \underline{f(r)=\frac{1}{6}r^2-\frac{C_1}{r}+C_2}.$$

$\lim_{r\to 0}|f(r)|<\infty \Rightarrow \underline{C_1\equiv 0} \Rightarrow f(r)=\frac{1}{6}r^2+C_2$. (*)

$f(R)=0 \Rightarrow \frac{1}{6}R^2+C_2=0 \Leftrightarrow \underline{C_2=-\frac{1}{6}R^2}$.

<u>Resultat</u>: $u(x,y,z)=\frac{1}{6}(x^2+y^2+z^2-R^2)$.

<u>Övning 2.77 (s. 37)</u>

$\underline{u(x_1,x_2,x_3,\dots,x_n)=f(r),\ r=(x^2+y^2+z^2)^{1/2}}$.

$r^2 = x_1^2+x_2^2+\dots+x_n^2 \Rightarrow 2r\frac{\partial r}{\partial x_i} = 2x_i \Leftrightarrow \underline{\frac{\partial r}{\partial x_i} = \frac{x_i}{r}}$ $(i=1,2,\dots,n)$.

$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} f(r) = f'(r)\frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{r} = x_i\cdot r^{-1} f'(r)$, $i=1,2,\dots,n$.

$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}(x_i r^{-1} f'(r)) = \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)f'(r) + \frac{x_i^2}{r^2}f''(r)$, $i=1,\dots,n$.

$\Delta u = \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = \left(\frac{n}{r} - \frac{1}{r^3}\sum_{i=1}^{n} x_i^2\right)f'(r) + \frac{1}{r^2}f''(r)\sum_{i=1}^{n} x_i^2 =$

$= \left(\frac{n}{r} - \frac{r^2}{r^3}\right)f'(r) + f''(r) = \underline{\frac{n-1}{r}f'(r) + f''(r) = 0} \Leftrightarrow$

(i) $\underline{n=2}$: $\frac{1}{r}f'(r) + f''(r) = 0 \Leftrightarrow rf''(r) + f'(r) = 0 \Leftrightarrow$

$\Leftrightarrow (rf'(r))' = 0 \Leftrightarrow rf'(r) = C_1 \Leftrightarrow f'(r) = \frac{C_1}{r} \Leftrightarrow$

$\Leftrightarrow \underline{f(r) = C_1 \ln r + C_2}$.

(ii) $\underline{n \neq 2}$: $\frac{n-1}{r}f'(r) + f''(r) = 0 \Leftrightarrow rf''(r) + (n-1)f'(r) = 0$

$\Leftrightarrow r^{n-1}f''(r) + (n-1)r^{n-2}f'(r) = 0 \Leftrightarrow$

$\Leftrightarrow (r^{n-1}f'(r))' = 0 \Leftrightarrow r^{n-1}\cdot f'(r) = C_1 \Leftrightarrow$

$\Leftrightarrow f'(r) = C_1 r^{1-n} \Leftrightarrow \underline{f(r) = \frac{C_1}{2-n}r^{2-n} + C_2}$.

Övning 2.78 (s.37)

$\underline{x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = 0,\ f(x,0) = x^4}$.

$\begin{cases} x = r\cos v \\ y = r\sin v \end{cases} \Leftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \wedge \frac{\partial r}{\partial y} = \frac{y}{r} \\ v = \arctan\frac{y}{x} \Rightarrow \frac{\partial v}{\partial x} = -\frac{y}{r^2} \wedge \frac{\partial v}{\partial y} = \frac{x}{r^2} \end{cases}$

$f(r\cos v, r\sin v) = \tilde{f}(r,v) \Rightarrow x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x} = x\left(\frac{\partial \tilde{f}}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \tilde{f}}{\partial v}\frac{\partial v}{\partial y}\right) - y\left(\frac{\partial \tilde{f}}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \tilde{f}}{\partial v}\frac{\partial v}{\partial x}\right) = \left(x\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial x}\right)\frac{\partial \tilde{f}}{\partial r} +$

$+ \left(x\frac{\partial v}{\partial y} - y\frac{\partial v}{\partial x}\right)\frac{\partial \tilde{f}}{\partial v} = \frac{xy-yx}{r}\frac{\partial \tilde{f}}{\partial u} + \frac{x^2+y^2}{r^2}\frac{\partial \tilde{f}}{\partial v} = \frac{\partial \tilde{f}}{\partial v} = 0 \Leftrightarrow$

$\Leftrightarrow \tilde{f}(r,v) = \phi(r) \Leftrightarrow f(x,y) = \phi(\sqrt{x^2+y^2})$;

$f(x,0) = x^4 \Rightarrow \phi(x) = x^4 \Leftrightarrow \phi(u) = u^4 \Leftrightarrow \phi(\sqrt{x^2+y^2}) =$

$= (\sqrt{x^2+y^2})^4 = (x^2+y^2)^2 \Rightarrow \underline{f(x,y) = (x^2+y^2)^2}$.

Övning 2.79 (s.37)

$\underline{u(x,y) = e^{ax+by}f(x,y)}$; $\underline{\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x^2}}$; $\underline{\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial f}{\partial x}}$. (*)

$\frac{\partial u}{\partial x} = au(x,y) + e^{ax+by}\frac{\partial f}{\partial x}$;

$\frac{\partial u}{\partial y} = bu(x,y) + e^{ax+by}\frac{\partial f}{\partial y}$;

$\frac{\partial^2 u}{\partial x^2} = a^2u(x,y) + 2ae^{ax+by}\frac{\partial f}{\partial x} + e^{ax+by}\frac{\partial^2 f}{\partial x^2}$;

$VL = \frac{\partial u}{\partial y} = bu(x,y) + e^{ax+by}\frac{\partial f}{\partial y} = a^2u(xy) + 2a\, e^{ax+by}\frac{\partial f}{\partial x}$

$+ e^{ax+by}\frac{\partial^2 f}{\partial x^2} = HL \overset{(*)}{\Leftrightarrow} bf(x,y) + \frac{\partial^2 f}{\partial x^2} - 2\frac{\partial f}{\partial x} =$

$= a^2f(x,y) + 2a\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} \Leftrightarrow (a^2-b)f(x,y) + 2(a+1)\frac{\partial f}{\partial x} = 0$

$\Rightarrow a^2-b = 0 \wedge a+1 = 0 \Leftrightarrow \underline{a=-1 \wedge b=1}$

Övning 2.80 (s. 37)

$\underline{\Pi: x+2y+z=3}$, $\underline{C: x^2+y^2-z^2=1}$

forts.

Antag att ytorna tangerar varandra i (a,b,c).
C är en nivåyta till $f(x,y,z)=x^2+y^2-z^2$.
I "tangeringspunkten" (a,b,c) har ytorna gemensam normal, vilket ger sambandet
$\operatorname{grad} f(a,b,c)=(2a,2b,-2c)=k\cdot(1,2,1)\ ,\quad k\neq 0.$
$\Leftrightarrow (a,b,-c)=\frac{k}{2}(1,2,1)\in\pi \Rightarrow \frac{k}{2}+2\cdot k-\frac{k}{2}=3 \Leftrightarrow k=\frac{3}{2}.$
$\Rightarrow (a,b,c)=(\frac{3}{2},3,-\frac{3}{2}) \Rightarrow f(\frac{3}{2},3,-\frac{3}{2})=9\neq 1$;
Resultat: Ytorna tangerar inte varandra.

Övning 2.81 (s.37)

$f(x,y)=x^2+y^2+x^3+y^3$

$\frac{\partial f}{\partial x}=2x+3x^2,\ \frac{\partial f}{\partial y}=2y+3y^2;\ \frac{\partial^2 f}{\partial x^2}=2+6x,\ \frac{\partial^2 f}{\partial y^2}=2+6y,$
$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=0.$

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}\Leftrightarrow\begin{cases}2x+3x^2=0\\2y+3y^2=0\end{cases}\Leftrightarrow\begin{cases}3x(x+\frac{2}{3})=0\\3y(y+\frac{2}{3})=0\end{cases}\Leftrightarrow\begin{cases}x=0\\y=0\end{cases}\vee$$
$$\vee\begin{cases}x=0\\y=-\frac{2}{3}\end{cases}\vee\begin{cases}x=-2/3\\y=0\end{cases}\vee\begin{cases}x=-2/3\\y=-2/3\end{cases}.$$

f:s stationära punkter är $(0,0),(0,-\frac{2}{3}),(-\frac{2}{3},0)$ och $(-\frac{2}{3},-\frac{2}{3})$. Det gäller att sortera dem.

(i) $\underline{(x,y)=(0,0)} \Rightarrow f''_{xx}(0,0)=f''_{yy}(0,0)=2,\ f''_{xy}(0,0)=0$;
$Q(h,k)=2h^2+2k^2$ pos. definit $\Rightarrow$ $(0,0)$ minpkt.

(ii) $(x,y)=(0,-\frac{2}{3}) \Rightarrow f''_{xx}=2 \wedge f''_{yy}=-2.\wedge f''_{xy}=0$;
$Q(h,k)=2h^2-2k^2$ indefinit $\Rightarrow$ $(0,-2/3)$ sadelpunkt.

(iii) $(x,y)=(-\frac{2}{3},0) \Rightarrow f''_{xx}=-2 \wedge f''_{yy}=2 \wedge f''_{xy}=0$;
$Q(h,k)=-2h^2+2k^2$ indefinit $\Rightarrow$ $(-\frac{2}{3},0)$ sadelpunkt.

(iv) $(x,y)=(-\frac{2}{3},-\frac{2}{3}) \Rightarrow f''_{xx}=-2 \wedge f''_{yy}=-2 \wedge f''_{yx}=0$;
$Q(h,k)=-2h^2-2k^2$ negativ definit $\Rightarrow (-\frac{2}{3},-\frac{2}{3})$ maximipunkt.

Resultat: f uppvisar ett lokalt minimum i origo och ett lokalt maximum i $(-\frac{2}{3},-\frac{2}{3})$.

Övning 2.82 (s.38)

a) $f:\mathbb{R}^2\to\mathbb{R},\ f\in C^1,\ a\in D_f.$
Låt $\hat{v}$ vara en en enhetsvektor parallell med en vektor v.

$\frac{\partial f}{\partial v}=f'_v(a)=\lim_{t\to 0}\frac{f(a+t\hat{v})-f(a)}{t}=\lim_{t\to 0}\frac{d}{dt}f(a+t\hat{v})=$
$=\lim_{t\to 0}\operatorname{grad} f(a+t\hat{v})\cdot\hat{v}=\operatorname{grad} f(a)\cdot\hat{v}.$

b) $f(x,y)=x^2y^3$

På en linje kan vi förflytta oss i två riktningar, fram och tillbaka.

$\operatorname{grad} f(x,y)=(2xy^3, 3x^2y^2) \Rightarrow \operatorname{grad} f(2,1)=(4,12)$. (*)

$l: 3x+4y=10 \Rightarrow (2,1)\in l$.

$v=(-4,3)$ är en riktningsvektor till l.

$\hat{v}_1=\frac{v}{|v|}=(-\frac{4}{5},\frac{3}{5})$ eller $\hat{v}_2=(\frac{4}{5},-\frac{3}{5})$.

$f'_{v_1}(a)=f'_{v_1}(2,1)=\operatorname{grad} f(2,1)\cdot(-\frac{4}{5},\frac{3}{5})\overset{(*)}{=}4$.

$f'_{v_2}(a)=f'_{v_2}(2,1)=\operatorname{grad} f(2,1)\cdot(\frac{4}{5},-\frac{3}{5})\overset{(*)}{=}-4$.

Resultat: Vi ska röra oss i riktningen $\hat{v}=(-\frac{4}{5},\frac{3}{5})$; ökningen sker då med faktorn 4.

Övning 2.83 (s. 38)

$y\frac{\partial^2 u}{\partial y\partial x}+x\frac{\partial^2 u}{\partial y^2}=0$; $u(x,y)=y\cdot f(v)$, $v=xy$.

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}yf(v)=y\frac{\partial}{\partial x}f(v)=yf'(v)\frac{\partial v}{\partial x}=y^2f'(v)$;

$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}yf(v)=f(v)+yf'(v)\frac{\partial v}{\partial y}=xyf'(v)+f(v)$

$\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial x})=\frac{\partial}{\partial y}(y^2f'(v))=2yf'(v)+xy^2f''(v)$;

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})=\frac{\partial}{\partial y}(xyf'(v)+f(v))=2xf'(v)+x^2yf''(v)$.

$VL=y\frac{\partial^2 u}{\partial y\partial x}+x\frac{\partial^2 u}{\partial y^2}=y(2yf'(v)+xy^2f''(v))+$
$+x(2xf'(v)+x^2yf''(v))=2(x^2+y^2)f'(v)+$
$+xy(x^2+y^2)f''(v)=0 \Leftrightarrow 2f'(v)+vf''(v)=0\Leftrightarrow$
$\Leftrightarrow v^2f''(v)+2vf'(v)=0\Leftrightarrow (v^2f'(v))'=0\Leftrightarrow$
$\Leftrightarrow v^2f'(v)=C_1\Leftrightarrow f'(v)=\frac{C_1}{v^2}\Leftrightarrow f(v)=-\frac{C_1}{v}+C_2$

Resultat: Den sökta funktionen är $f(t)=\frac{A}{t}+B$.

Övning 2.84 (s. 38)

$f(x,y)=e^{x^2+y^2}(x+ay)$;

$\frac{\partial f}{\partial x}=e^{x^2+y^2}(1)+2xe^{x^2+y^2}(x+ay)=(1+2x^2+2axy)e^{x^2+y^2}$

$\frac{\partial f}{\partial y}=e^{x^2+y^2}\cdot a+2ye^{x^2+y^2}(x+ay)=(a+2xy+2ay^2)e^{x^2+y^2}$

$\operatorname{grad} f(x,y)=((1+2x^2+2axy)e^{x^2+y^2}, (a+2xy+2ay^2)e^{x^2+y^2})$

$\operatorname{grad} f(1,1)=(e^2(3+2a), e^2(2+3a))$.

$\frac{\partial f}{\partial v}=f'_v(1,1)=(e^2(3+2a), e^2(2+3a))\cdot(\frac{3}{5},\frac{4}{5})=$
$=\frac{3}{5}e^2(3+2a)+\frac{4}{5}e^2(2+3a)=\frac{e^2}{5}(9+6a+8+12a)=$
$=\frac{e^2}{5}(18a+17)$.

$f'_v(1,1)$ är maximal då $\operatorname{grad} f(1,1)\parallel(\frac{3}{5},\frac{4}{5})\Leftrightarrow$

$\Leftrightarrow \frac{e^2(3+2a)}{3/5}=\frac{e^2(2+3a)}{4/5}\Leftrightarrow 4(3+2a)=3(2+3a)\Leftrightarrow$

$\Leftrightarrow 12+8a=6+9a \Leftrightarrow a=6 \Rightarrow \{f'_v(1,1)\}_{ma}=25e^2.$

Svar: $a=6$.

Övning 2.85 (s. 38)

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=xy;\quad u=x,\ v=\frac{x}{y}.$$

a) $$\frac{\partial f}{\partial x}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial \tilde f}{\partial u}+\frac{1}{y}\frac{\partial \tilde f}{\partial v}=\left(\frac{\partial}{\partial u}+\frac{v}{u}\frac{\partial}{\partial v}\right)\tilde f(u,v).$$

$$\frac{\partial f}{\partial y}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}=-\frac{x}{y^2}\frac{\partial \tilde f}{\partial v}=-\frac{v^2}{u}\frac{\partial}{\partial v}\tilde f(u,v).$$

Lägg märke till differentialoperatorerna

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial u}+\frac{v}{u}\frac{\partial}{\partial v},\quad \frac{\partial}{\partial v}=-\frac{v^2}{u}\frac{\partial}{\partial v};$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{v}{u}\frac{\partial}{\partial v}\right)\left(\frac{\partial \tilde f}{\partial u}+\frac{v}{u}\frac{\partial \tilde f}{\partial v}\right)=$$
$$=\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial u}+\frac{v}{u}\frac{\partial \tilde f}{\partial v}\right)+\frac{v}{u}\frac{\partial}{\partial v}\left(\frac{\partial \tilde f}{\partial u}+\frac{v}{u}\frac{\partial \tilde f}{\partial v}\right)=$$
$$=\frac{\partial^2 \tilde f}{\partial u^2}+\frac{v}{u^2}\frac{\partial \tilde f}{\partial v}+\frac{v}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{v}{u}\left(\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{1}{u}\frac{\partial \tilde f}{\partial v}+\frac{v}{u}\frac{\partial^2 \tilde f}{\partial v^2}\right)=$$
$$=\frac{\partial^2 \tilde f}{\partial u^2}+2\frac{v}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{v^2}{u^2}\frac{\partial^2 \tilde f}{\partial v^2};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=-\frac{v^2}{u}\frac{\partial}{\partial v}\left(-\frac{v^2}{u}\frac{\partial \tilde f}{\partial v}\right)=-\frac{v^2}{u}\left(-2\frac{v}{u}\frac{\partial \tilde f}{\partial v}-\right.$$
$$\left.-\frac{v^2}{u}\frac{\partial^2 \tilde f}{\partial v^2}\right)=4\frac{v^3}{u^2}\frac{\partial \tilde f}{\partial v}-\frac{v^4}{u^2}\frac{\partial^2 \tilde f}{\partial v^2}.$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{v}{u}\frac{\partial}{\partial v}\right)\left(-\frac{v^2}{u}\frac{\partial \tilde f}{\partial v}\right)=\frac{\partial}{\partial u}\left(-\frac{v^2}{u}\frac{\partial \tilde f}{\partial v}\right)+$$
$$+\frac{v}{u}\frac{\partial}{\partial v}\left(-\frac{v^2}{u}\frac{\partial \tilde f}{\partial v}\right)=\frac{v^2}{u^2}\frac{\partial \tilde f}{\partial v}-\frac{v^2}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}-2\frac{v^2}{u^2}\frac{\partial \tilde f}{\partial v}-$$
$$-\frac{v^3}{u^2}\frac{\partial^2 \tilde f}{\partial v^2}=-\frac{v^2}{u^2}\frac{\partial \tilde f}{\partial v}-\frac{v^2}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}-\frac{v^3}{u^2}\frac{\partial^2 \tilde f}{\partial v^2};$$

$$VL=x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=u^2\frac{\partial^2 f}{\partial x^2}+2\frac{u^2}{v}\frac{\partial^2 f}{\partial x\partial y}+$$
$$+\frac{u^2}{v^2}\frac{\partial^2 f}{\partial x^2}=u^2\left(\frac{\partial^2 \tilde f}{\partial u^2}+2\frac{v}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}+\frac{v^2}{u^2}\frac{\partial^2 f}{\partial v^2}\right)+$$
$$+2\frac{u^2}{v}\left(-\frac{v^2}{u^2}\frac{\partial \tilde f}{\partial v}-\frac{v^2}{u}\frac{\partial^2 \tilde f}{\partial u\partial v}-\frac{v^3}{u^2}\frac{\partial^2 \tilde f}{\partial v^2}\right)+\frac{u^2}{v^2}\left(2\frac{u^3}{u^2}\frac{\partial \tilde f}{\partial v}-\right.$$
$$\left.-\frac{v^4}{u^2}\frac{\partial^2 \tilde f}{\partial v^2}\right)=u^2\frac{\partial^2 \tilde f}{\partial u^2}+2uv\frac{\partial^2 \tilde f}{\partial u\partial v}+v^2\frac{\partial^2 \tilde f}{\partial v^2}-$$
$$-2v\frac{\partial \tilde f}{\partial v}-2uv\frac{\partial^2 \tilde f}{\partial u\partial v}-2v^2\frac{\partial^2 \tilde f}{\partial v^2}+2v\frac{\partial \tilde f}{\partial v}+v^2\frac{\partial^2 \tilde f}{\partial v^2}=$$
$$=u^2\frac{\partial^2 \tilde f}{\partial u^2}=xy=\frac{u^2}{v}=HL\Leftrightarrow\frac{\partial^2 \tilde f}{\partial u^2}=\frac{1}{v}.$$

b) $$\frac{\partial^2 \tilde f}{\partial u^2}=\frac{1}{v}\Rightarrow\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial u}\right)=\frac{1}{v}\Rightarrow\frac{\partial \tilde f}{\partial u}=\frac{u}{v}+\phi(v)\Leftrightarrow\tilde f(u,v)=$$
$$=\frac{u^2}{2v}+F(u)+u\cdot G(v)+H(v).$$

Resultat: $f(x,y)=\frac{1}{2}xy+xG\left(\frac{x}{y}\right)+H\left(\frac{x}{y}\right).$

Övning 2.86 (s. 38)

$\pi: 2x+2y+z=C,\quad S: x+y^2+z^4=1.$

C är nivåyta till funktionen $f(\mathbf{x})=x+y^2+z^4$.

Normalvektorn $n=(2,2,1)$ till π skall vara parallell med gradienten till f i tangeringspunkten, (α,β,γ) kalla den.

$\operatorname{grad} f(\mathbf{x})=(1,2y,4z^3)\Rightarrow \operatorname{grad} f(\alpha,\beta,\gamma)=(1,2\beta,4\gamma^3)$.

$\operatorname{grad} f(\alpha,\beta,\gamma)\parallel n\Rightarrow(1,2\beta,4\gamma^3)=k\cdot(2,2,1)\Rightarrow$

$\Rightarrow$ (efter identifikation) $\Rightarrow$ $k=\beta=\gamma=1/2$.

$f(\alpha,\frac{1}{2},\frac{1}{2})=1 \Rightarrow \alpha+\frac{1}{4}+\frac{1}{16}=1 \Leftrightarrow \alpha=\frac{11}{16}$.

Insättning av $(\frac{11}{16},\frac{1}{2},\frac{1}{2})$ i planets ekvation ger

$$C=2\cdot\frac{11}{16}+2\cdot\frac{1}{2}+\frac{1}{2}=\frac{11}{8}+\frac{3}{2}=\frac{23}{8}.$$

Övning 2.87 (s.38)

Enhetssfären är nivåyta till $f(x)=x^2+y^2+z^2$. Kalla tangeringspunkten (a,b,c). Tangentplanets ekvation i kombination med relationen $a^2+b^2+c^2=1$ ger

$\operatorname{grad} f(a,b,c)(x-a,y-b,z-c)=(2a,2b,2c)\cdot(x-a,y-b,z-c)=2ax+2by+2cz-2(a^2+b^2+c^2)=0 \Leftrightarrow$

$\pi:\ ax+by+cz=1.$

$$\begin{cases}(3,0,0)\in\pi \Rightarrow 3a=1\Leftrightarrow a=\frac{1}{3}\\ (0,3,0)\in\pi \Rightarrow 3b=1\Leftrightarrow b=\frac{1}{3}\end{cases} \Rightarrow c^2=1-\frac{2}{9} \Rightarrow c=\pm\frac{\sqrt{7}}{3}.$$

Resultat: Det finns två tangeringspunkter, nämligen $(\frac{1}{3},\frac{1}{3},\frac{\sqrt{7}}{3})$ och $(\frac{1}{3},\frac{1}{3},-\frac{\sqrt{7}}{3})$.

Övning 2.88 (s.38)

Se nästa sida.

$x\frac{\partial^2 f}{\partial x^2}-2y\frac{\partial^2 f}{\partial x\partial y}=0;\ f(x,y)=g(u),\ u=x^2y.$

$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}g(u)=g'(u)\frac{\partial u}{\partial x}=2xy\,g'(u)$

$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(2xyg'(u))=2yg'(u)+4x^2y^2g''(u);$

$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial y}(2xyg'(u))=2xg'(u)+2x^3yg''(u)$

$VL=x\frac{\partial^2 f}{\partial x^2}-2y\frac{\partial^2 f}{\partial y^2}=x(2yg'(u)+4x^2y^2g''(u))-$

$-2y(2xg'(u)+2x^3yg''(u))=2xyg'(u)+4x^3y^2g''(u)$

$-4xyg'(u)-4x^3y^2g''(u)=-2xyg'(u)=x^3y^2=HL$

$\Leftrightarrow -2g'(u)=x^2y=u \Leftrightarrow g'(u)=-\frac{1}{2}u \Leftrightarrow g(u)=-\frac{u^2}{4}+C.$

Svar: $f(x,y)=-\frac{1}{4}x^4y^2+C.$

Övning 2.89 (s.38)

$v\frac{\partial Q}{\partial v}-p\frac{\partial Q}{\partial p}=f(pv)$, $Q(p,v)=\tilde{Q}(x,y)$, $x=pv, y=p$.

$$\begin{cases}\frac{\partial Q}{\partial v}=\frac{\partial\tilde{Q}}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial\tilde{Q}}{\partial y}\frac{\partial y}{\partial v}=p\frac{\partial\tilde{Q}}{\partial x}\\ \frac{\partial Q}{\partial p}=\frac{\partial\tilde{Q}}{\partial x}\frac{\partial x}{\partial p}+\frac{\partial\tilde{Q}}{\partial y}\frac{\partial y}{\partial p}=v\frac{\partial\tilde{Q}}{\partial x}-\frac{\partial\tilde{Q}}{\partial y}\end{cases} \Rightarrow VL=v\frac{\partial Q}{\partial v}-p\frac{\partial Q}{\partial p}=$

$=-p\frac{\partial\tilde{Q}}{\partial y}=f(pv)=f(x)=HL \Leftrightarrow -y\frac{\partial Q}{\partial y}=f(x) \Leftrightarrow$

$\Leftrightarrow \frac{\partial\tilde{Q}}{\partial y}=-\frac{1}{y}f(x) \Leftrightarrow \tilde{Q}(x,y)=g(x)-f(x)\ln y \Rightarrow$

$\Rightarrow \underline{Q(p,v)=g(pv)-f(pv)\ln p}.$

Anm. g är en godtycklig C^1-funktion i $\mathbb{R}$.

Övning 2.90 (s. 39)

a) $f(x,y)=\bar{f}(r,\theta)$; $x=r\cos\theta$, $y=r\sin\theta$.

$$\frac{\partial\bar{f}}{\partial\theta}=\frac{\partial}{\partial\theta}f(x,y)=\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta}=-r\sin\theta\frac{\partial f}{\partial x}+$$
$$+r\cos\theta\frac{\partial f}{\partial y}=-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}.$$

Anm. $\frac{\partial}{\partial\theta}=x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}$.

b) $VL=y\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial y}=-\frac{\partial\bar{f}}{\partial\theta}=r(\sin\theta+\cos\theta)=x+y=HL$

$\Leftrightarrow \frac{\partial\bar{f}}{\partial\theta}=-r(\sin\theta+\cos\theta)\Leftrightarrow \bar{f}(r,\theta)=-r(\sin\theta-\cos\theta)+$

$+\phi(r)\Leftrightarrow \bar{f}(r,\theta)=r\cos\theta-r\sin\theta+\phi(r)\Leftrightarrow$

$\Leftrightarrow \underline{f(x,y)=x-y+\phi(\sqrt{x^2+y^2})}$, $\phi\in C^1$.

Övning 2.91 (s. 39)

$f(x,y)=x^2+2xy+xy^2$.

$\frac{\partial f}{\partial x}=2x+2y+y^2$, $\frac{\partial f}{\partial y}=2x+2xy$;

$\frac{\partial^2 f}{\partial x^2}=2$, $\frac{\partial^2 f}{\partial x\partial y}=2+2y$, $\frac{\partial^2 f}{\partial y^2}=2x$.

Kritiska (stationära) punkter

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0\Rightarrow\begin{cases}2x+2y+y^2=0\\2x+2xy=0\end{cases}\Leftrightarrow\begin{cases}2x+2y+y^2=0\\x=0\vee y=-1\end{cases};$$

$\underline{x=0}\Rightarrow 2y+y^2=0\Leftrightarrow \underline{y=0\vee y=-2}$.

$y=-1\Rightarrow 2x-1=0\Leftrightarrow x=1/2$.

Vi har tre stationära punkter, nämligen $(0,0)$, $(0,-2)$ och $(\frac{1}{2},-1)$.

Extrempunkter

$\underline{(x,y)=(0,0)}\Rightarrow f''_{xx}=2\wedge f''_{xy}=2\wedge f''_{yy}=0\Rightarrow Q(h,k)=$
$=2h^2+4hk$ indefinit $\Rightarrow (0,0)$ sadelpunkt.

$\underline{(x,y)=(\frac{1}{2},-1)}\Rightarrow f''_{xx}=2\wedge f''_{xy}=0\wedge f''_{yy}=1\Rightarrow Q(h,k)=$
$=2h^2+k^2$ pos. definit $\Rightarrow (\frac{1}{2},-1)$ minimipunkt.

$(x,y)=(0,-2)\Rightarrow f''_{xx}=2\wedge f''_{xy}=-2\wedge f''_{yy}=-4\Rightarrow$
$\Rightarrow Q(h,k)=2h^2-8hk-2k^2=2(h-k)^2-10k^2$, som är indefinit $\Rightarrow (0,-2)$ sadelpunkt.

Resultat: Den enda extrempunkten ä $(\frac{1}{2},-1)$. Denna är en (lokal) minimipunkt.

Övning 2.92 (s. 39)

$D=x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$; $z(x,y)=w(u,v)$;

$$\begin{cases}x=e^{u+v}\\y=e^{u-v}\end{cases}\Leftrightarrow\begin{cases}u+v=\ln x\\u-v=\ln y\end{cases}\Leftrightarrow\begin{cases}u(x,y)=\frac{1}{2}\ln x+\frac{1}{2}\ln y\\v(x,y)=\frac{1}{2}\ln x-\frac{1}{2}\ln y\end{cases};$$

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}w=\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}=\frac{1}{2x}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)w\,;$$
$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}w=\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{2y}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial w}\right)w\,;$$

Låt oss införa differentialoperatorerna

$$(*)\quad \frac{\partial}{\partial x}=\frac{1}{2}e^{-u-v}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right),\ \frac{\partial}{\partial y}=\frac{1}{2}e^{-u+v}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right).$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{1}{2}e^{-u-v}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\frac{1}{2}e^{-u-v}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)=$$
$$=\frac{1}{4}e^{-u-v}\left(\frac{\partial}{\partial u}\left(e^{-u-v}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)\right)+\frac{\partial}{\partial v}\left(e^{-u-v}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)\right)\right)=$$
$$=\frac{1}{4}e^{-u-v}\Big[-e^{-u-v}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)+e^{-u-v}\left(\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\partial v}\right)+$$
$$+-e^{-u-v}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)+e^{-u-v}\left(\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}\right)\Big]=$$
$$=\frac{1}{4}e^{-2u-2v}\left(-\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}+\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial u\partial v}-\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}+\right.$$
$$\left.+\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}\right)=$$
$$=\frac{1}{4}e^{-2u-2v}\left(\frac{\partial^2 w}{\partial u^2}+2\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}-2\frac{\partial w}{\partial u}-2\frac{\partial w}{\partial v}\right);$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{1}{2}e^{-u+v}\left(\frac{\partial}{\partial u}-\frac{\partial}{\partial v}\right)\frac{1}{2}e^{-u+v}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)=$$
$$=\frac{1}{4}e^{-u+v}\left(\frac{\partial}{\partial u}e^{-u+v}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)-\frac{\partial}{\partial v}e^{-u+v}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)\right)=$$
$$=\frac{1}{4}e^{-u+v}\left(-e^{-u+v}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)+e^{-u+v}\left(\frac{\partial^2 w}{\partial u^2}-\frac{\partial^2 w}{\partial u\partial v}\right)-\right.$$
$$\left.-e^{-u+v}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)-e^{-u+v}\left(\frac{\partial^2 w}{\partial u\partial v}-\frac{\partial^2 w}{\partial v^2}\right)\right)=$$
$$=\frac{1}{4}e^{-2u+2v}\left(-\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}+\frac{\partial^2 w}{\partial u^2}-\frac{\partial^2 w}{\partial u\partial v}-\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}-\right.$$
$$\left.-\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}\right)=$$
$$=\frac{1}{4}e^{-2u+2v}\left(\frac{\partial^2 w}{\partial u^2}-2\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}-2\frac{\partial w}{\partial w}+2\frac{\partial w}{\partial v}\right);$$

$$D=x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=$$
$$=\frac{1}{4}\left(\frac{\partial^2 w}{\partial u^2}+2\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}-\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)-$$
$$-\frac{1}{4}\left(\frac{\partial^2 w}{\partial u^2}-2\frac{\partial^2 w}{\partial u\partial v}+\frac{\partial^2 w}{\partial v^2}-\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)+$$
$$+\frac{1}{2}\left(\frac{\partial w}{\partial u}+\frac{\partial w}{\partial v}\right)-\frac{1}{2}\left(\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}\right)=$$
$$=\frac{\partial^2 w}{\partial u\partial v}.$$

Övning 2.93 (s. 39)

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}=\frac{R_1+R_2}{R_1R_2}\Leftrightarrow R=\frac{R_1R_2}{R_1+R_2}\,;$$
$$dR=\frac{\partial R}{\partial R_1}dR_1+\frac{\partial R}{\partial R_2}dR_2=\left(\frac{R_2}{R_1+R_2}\right)^2dR_1+\left(\frac{R_1}{R_1+R_2}\right)^2dR_2\,;$$
$$\Delta R=dR=\pm\left(\left(\frac{300}{300+200}\right)^2\cdot 0{,}5+\left(\frac{200}{300+200}\right)^2\cdot 1\right)=\pm 0{,}34.$$
$$R=\frac{R_1R_2}{R_1+R_2}=\frac{300\cdot 200}{300+200}=120.$$

Resultat: $R=(120\pm 0{,}34)\,\Omega$.

Övning 2.94 (s.39)

$$f(x,y)=e^y\left(y+1-(y-1)\sin x\right).$$
$$\frac{\partial f}{\partial x}=-e^y(y-1)\cos x\,,\quad \frac{\partial f}{\partial y}=e^y(2+y-y\sin x).$$
$$\frac{\partial^2 f}{\partial x^2}=e^y(y-1)\sin x,\quad \frac{\partial^2 f}{\partial x\partial y}=e^y(y-1)\sin x,$$
$$\frac{\partial^2 f}{\partial y^2}=e^y(3+y+(y-1)\sin x).$$

Kritiska (stationära) punkter söks.

$$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y}\Leftrightarrow\begin{cases}e^y(y-1)\cos x=0\\ e^y(2+y-\sin x\cdot y)=0\end{cases}\Leftrightarrow\begin{cases}y=1\vee\cos x=0\\ 2+y-y\sin x=0\end{cases}$$

$$\Leftrightarrow\begin{cases}y=1\\ 2+y(1-\sin x)=0\end{cases}\vee\begin{cases}\cos x=0\\ 2+y(1-\sin x)=0\end{cases}\Leftrightarrow\begin{cases}\cos x=0\\ 2+y(1-\sin x)=0\end{cases}$$

$$\Leftrightarrow x=\frac{\pi}{2}+n\pi\wedge y=\frac{2}{\sin x-1}\Leftrightarrow x=\frac{\pi}{2}+n\pi\wedge y=\frac{2}{(-1)^n-1}$$

$\Leftrightarrow$ (n ska vara udda) $\Leftrightarrow x=(2k+\frac{3}{2})\pi\wedge y=-1\Leftrightarrow$

$\Leftrightarrow (x,y)=((2k+\frac{3}{2})\pi,-1)\Rightarrow f''_{xx}=\frac{2}{e}\wedge f''_{xy}=\frac{2}{e}\wedge f''_{yy}=\frac{4}{e}$

$\Rightarrow Q(h,k)=\frac{2}{e}(h^2+2hk+2k^2)=\frac{2}{e}((h+k)^2+k^2)$ pos. definit $\Rightarrow\{((2k+\frac{3}{2})\pi,-1);\ k\in\mathbb{Z}\}$ <u>minimipunkter</u>.

Differentialkalkyl för vektorvärda funktioner

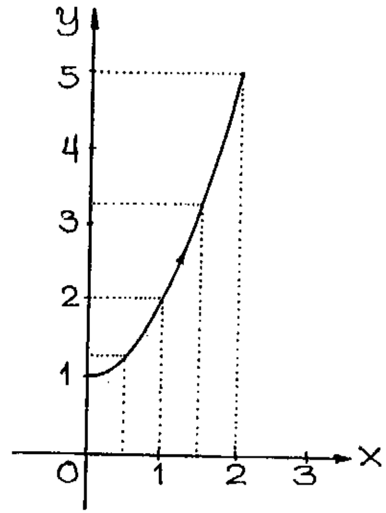
Kurvor och ytor

Övning 3.1 (s.61)

a) $\mathbf{x}(t)=(t,t^2+1),\quad 0\le t\le 2$

$$\mathbf{x}(t)=(x(t),y(t))=(t,t^2+1)\Leftrightarrow\begin{cases}x(t)=t\\ y(t)=t^2+1\end{cases};$$

t	0	0,5	1	1,5	2
$x(t)$	0	0,5	1	1,5	2
$y(t)$	0	1,25	2	3,25	5



<u>Anm.</u> $x=t\wedge y=t^2+1\Rightarrow y=x^2+1$.

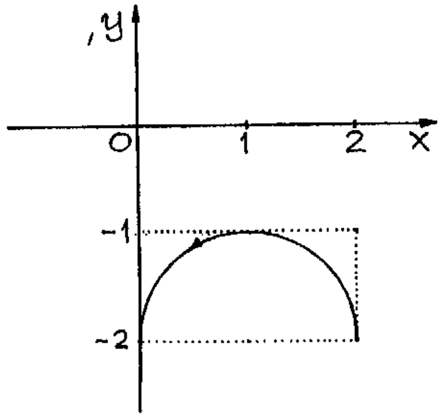
Obs! Enkel pil! Genom att eliminera parametern förlorar vi riktningen.

b) $\mathbf{x}(t) = (1+\cos t, -2+\sin t),\ 0 \le t \le \pi.$

$$\mathbf{x}(t) = (x(t), y(t)) = (1+\cos t, -2+\sin t) \Leftrightarrow \begin{cases} x(t) = 1+\cos t \\ y(t) = -2+\sin t \end{cases}$$

$$\Leftrightarrow \begin{cases} x-1 = \cos t \\ y+2 = \sin t \end{cases} \Rightarrow (x-1)^2 + (y+2)^2 = \cos^2 t + \sin^2 t = 1.$$

$$0 \le t \le \pi \Rightarrow \begin{cases} -1 \le \cos t \le 1 \\ 0 \le \sin t \le 1 \end{cases} \Leftrightarrow \begin{cases} 0 \le 1+\cos t \le 2 \\ -2 \le -2+\sin t \le -1 \end{cases} \Leftrightarrow \begin{cases} 0 \le x \le 2 \\ -2 \le y \le -1 \end{cases}$$



Anm. När det gäller enkla kurvor som ovan, kan man hoppa över derivator och sånt. Riktningen är inget problem heller.

c) $\mathbf{x}(t) = (\cos t, \cos 2t),\ \pi \le t \le 2\pi.$

$y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$;

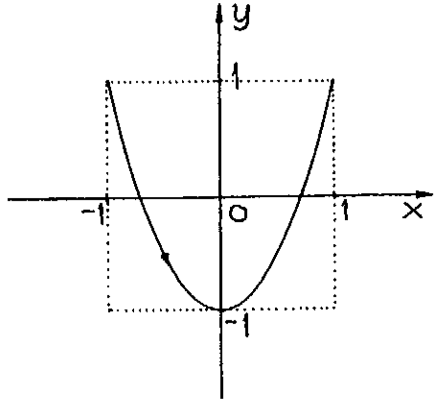
forts.

$\pi \le t \le 2\pi \Rightarrow -1 \le \cos t \le 1 \Leftrightarrow -1 \le x \le 1.$

Kurvans (eg. kurvbågens) värdemängd är

$$V = \{(x,y) : y = 2x^2 - 1,\ -1 \le x \le 1\}.$$

Begynnelsepunkten är $(-1,1)$ och slutpunkten $(1,1)$.



Lägg märke till att en kurvas värdemängd är bara bågen, alltså ingen riktning. Denna distinktion görs inte i gymnasiet och inte i envariabelkursen heller.

d) $\mathbf{x}(t) = (\cos t, \cos 2t),\ 0 \le t \le 2\pi.$

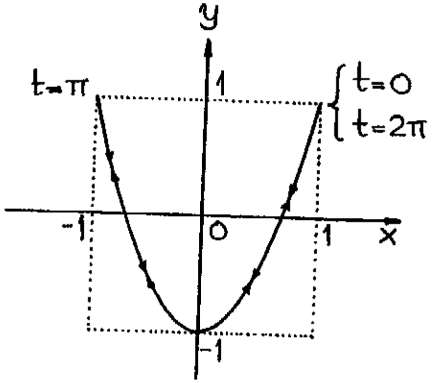
Vi delar parameterintervallet i två delar

$$[0, 2\pi] = [0, \pi] \cup [\pi, 2\pi].$$

Kurvan $\mathbf{x}(t) = (\cos t, \cos 2t)$, $\pi \le t \le 2\pi$, är identisk med kurvan i c) ovan. Kurvan

$$\mathbf{x}(t)=(\cos t, \cos 2t),\ 0\leq t\leq\pi,$$

har samma värdemängd som kurvan i c), alltså en parabelbåge, men riktningen är den motsatta. Den totala kurvan $\mathbf{x}(t) = (x(t), y(t))$, $0\leq t\leq 2\pi$ har således samma begynnelse- och slutpunkt, nämligen $(1,1)$.



Övning 3.2 (s.61)

a) $\mathbf{x}(t)=(x(t),y(t))=(t,t^2+1)$;

$\mathbf{x}'(t)=(x'(t),y'(t))=(1,2t)$;

$\mathbf{x}'(1)=(x'(1),y'(1))=\underline{(1,2)}.$

b) $\mathbf{x}(t)=(\cos t+1, \sin t-2)\Rightarrow \mathbf{x}'(t)=(-\sin t,\cos t)\Rightarrow$

$\Rightarrow \mathbf{x}'(\frac{\pi}{2})=(-\sin\frac{\pi}{2},\cos\frac{\pi}{2})=\underline{(-1,0)}.$

c) $\mathbf{x}(t)=(\cos t,\cos 2t)\Rightarrow \mathbf{x}'(t)=(-\sin t,-2\sin 2t)\Rightarrow$

$\Rightarrow \mathbf{x}'(\frac{3\pi}{2})=(-\sin\frac{3\pi}{2},-2\sin 3\pi)=\underline{(1,0)}.$

Övning 3.3 (s. 61)

$\underline{\mathbf{x}(t)=(\cos t, 2\sin t, t),\ t>0.}$

$\mathbf{x}(t)=(x(t),y(t),z(t))=(\cos t, 2\sin t, t)$; $\underline{z(\pi)=\pi.}$

a) $\mathbf{x}'(t)=(x'(t),y'(t),z'(t))=(-\sin t, 2\cos t, 1)$;

$\mathbf{x}'(\pi)=(x'(\pi),y'(\pi),z'(\pi))=\underline{(0,-2,1)}.$

b) $v(\pi)=|\mathbf{x}'(\pi)|=|(0,-2,1)|=\sqrt{0^2+2^2+1^2}=\underline{\sqrt{5}}.$

c) $\mathbf{x}''(t)=(x''(t),y''(t),z''(t))=(-\cos t,-2\sin t,0)$;

$\mathbf{x}''(\pi)=(x''(\pi),y''(\pi),z''(\pi))=\underline{(1,0,0)}.$

Övning 3.4 (s. 61)

Hyperboloiden $x^2+y^2-z^2=1$ är en nivåyta till funktionen $f(x,y,z)=x^2+y^2-z^2$; $f(2,1,2)=1$. $\operatorname{grad} f(x,y,z)=(2x,2y,-2z)\Rightarrow \operatorname{grad} f(2,1,2)=(4,2,-4)$. En normalvektor till planet $x-y+z=3$ är $\mathbf{n}=(1,-1,1)$. En riktningsvektor till skärningskurvan i punkten $(2,1,2)$ är parallell med

$$n \times \operatorname{grad} f(2,1,2) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 1 & -1 & 1 \\ 4 & 2 & -4 \end{vmatrix} = (2,8,6) = 2\cdot\underline{(1,4,3)}.$$

Tangentens ekvation i $(2,1,2)$ blir således

$$\underline{x(t) = (2,1,2) + t(1,4,3),\ t\in\mathbb{R}}.$$

Övning 3.5 (s. 61)

$\underline{\gamma : [0,2\pi] \to \mathbb{R}^3,\ t \mapsto (\cos t, \sin t, \cos 2t)}.$

$(x(t), y(t), z(t)) = (\cos t, \sin t, \cos 2t)$;

a) $x^2 - y^2 = \cos^2 t - \sin^2 t = \cos 2t = z$.

b) $(x'(t), y'(t), z'(t)) = (-\sin t, \cos t, -2\sin 2t)$

$$v = |(x'(t), y'(t), z'(t))| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} =$$
$$= \sqrt{\sin^2 t + \cos^2 t + 4\sin^2 2t} = \sqrt{1 + 4\sin^2 2t};$$

$$v_{max} = \left\{\sqrt{1+4\sin^2 2t}\right\}_{max} = \sqrt{1 + 4\{\sin^2 2t\}_{max}};$$

$0 \le t \le 2\pi \Rightarrow 0 \le \sin^2 2t \le 1 \Rightarrow \{\sin^2 2t\}_{max} = 1$.

$\sin^2 2t = 1 \Leftrightarrow \sin 2t = \pm 1 \Leftrightarrow 2t = (n+\frac{1}{2})\pi \Leftrightarrow \underline{t = (n+\frac{1}{2})\frac{\pi}{2}}$.

$0 \le t \le 2\pi \Rightarrow 0 \le (n+\frac{1}{2})\frac{\pi}{2} \le 2\pi \Leftrightarrow 0 \le n + \frac{1}{2} \le 4 \Leftrightarrow$

$\Leftrightarrow -\frac{1}{2} \le n \le \frac{7}{2} \Leftrightarrow \underline{n=0} \lor \underline{n=1} \lor \underline{n=2} \lor \underline{n=3} \Leftrightarrow$

$\Leftrightarrow t = \frac{\pi}{4} \lor t = \frac{3\pi}{4} \lor t = \frac{5\pi}{4} \lor t = \frac{7\pi}{4} \Leftrightarrow$ forts.

$\Leftrightarrow x = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \lor x = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) \lor x = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$
$\lor\ x = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$.

Resultat: Partikeln har störst fart i punkterna $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$ och $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$.

Övning 3.6 (s. 61)

$\underline{r(s,t) = ((2-\cos t)\cos s, (2-\cos t)\sin s, \sin t), -\pi \le s,t \le \pi}.$

$$r(s_0,t_0) = (1,\sqrt{3},1) \Rightarrow \begin{cases} (2-\cos t_0)\cos s_0 = 1 \\ (2-\cos t_0)\sin s_0 = \sqrt{3} \\ \sin t_0 = 1 \end{cases} \Leftrightarrow \begin{cases} 2\cos s_0 = 1 \\ 2\sin s_0 = \sqrt{3} \\ \sin t_0 = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cos s_0 = 1/2 \\ \sin s_0 = \sqrt{3}/2 \\ \sin t_0 = 1 \end{cases} \Leftrightarrow (s_0, t_0) = (\tfrac{\pi}{3}, \tfrac{\pi}{2}).$$

$$\begin{cases} r'_s(s,t) = ((\cos t - 2)\sin s, (2-\cos t)\cos s, 0) \\ r'_t(s,t) = (\sin t\cos s, \sin t \sin s, \cos t) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} r'_s(s_0,t_0) = r'_s(\frac{\pi}{3}, \frac{\pi}{2}) = (-\sqrt{3}, 1, 0) \\ r'_t(s_0,t_0) = r'_t(\frac{\pi}{3}, \frac{\pi}{2}) = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0) \end{cases};$$

En normalriktning i punkten $(1,\sqrt{3},1)$ ges av

$$n = r'_s(s_0,t_0) \times r'_t(s_0,t_0) = (-\sqrt{3},1,0)\times(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0) = \underline{(0,0,-2)}.$$

Anm. Hela teorin om normalriktning finns att läsa på s. 110-111 i grundboken.

Övning 3.7 (s. 62)

$\mathbf{r}(s,t) = (s\cdot\cos t, s\cdot\sin t, s^2),\quad 0<s<2,\ 0<t<2\pi.$

$$\mathbf{r}(s_0,t_0) = (0,1,1) \Rightarrow \begin{cases} s_0\cos t_0 = 0 \\ s_0\sin t_0 = 1 \\ s_0^2 = 1 \end{cases} \Leftrightarrow \begin{cases} \cos t_0 = 0 \\ \sin t_0 = 1 \\ s_0 = 1 \end{cases} \Leftrightarrow \begin{cases} s_0 = 1 \\ t_0 = \frac{\pi}{2} \end{cases};$$

$\mathbf{r}'_s(s,t) = (\cos t, \sin t, 2s)$, $\mathbf{r}'_t(s,t) = (-s\cdot\sin t, s\cos t, 0)$.

$\mathbf{r}'_s(1,\frac{\pi}{2}) = (0,1,2)$, $\mathbf{r}'_t(1,\frac{\pi}{2}) = (-1,0,0)$;

En normalriktning i punkten (0,1,1) ges av

$$\mathbf{n} = \mathbf{r}'_s(1,\tfrac{\pi}{2}) \times \mathbf{r}'_t(1,\tfrac{\pi}{2}) = (0,1,2)\times(-1,0,0) = (0,-2,1).$$

Anm. $x^2+y^2 = (s\cos t)^2 + (s\sin t)^2 = s^2 = z$. Ytan är en rotationsparaboloid med symmetriaxel z-axeln och minimipunkten (toppen) i origo (0,0,0).

Funktinaldeterminanter

Övning 3.8 (s. 62)

a) $$\begin{cases} y_1 = x_1^2 + 2x_2 \Rightarrow \frac{\partial y_1}{\partial x_1} = 2x_1 \wedge \frac{\partial y_1}{\partial x_2} = 2 \\ y_2 = x_1 + x_2 \Rightarrow \frac{\partial y_2}{\partial x_1} = 1 = \frac{\partial y_2}{\partial x_2} \end{cases} \Rightarrow f'(\mathbf{x}) = \begin{bmatrix} 2x_1 & 2 \\ 1 & 1 \end{bmatrix}.$$

Om funktionalmatriser finns det att läsa på sidan 112-113 i grundboken. Funktionalmatrisen ovan betecknas $\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}$.

b) $$\begin{cases} y_1 = 3x_1 + 2x_2 - 5 \Rightarrow \frac{\partial y_1}{\partial x_1} = 3 \wedge \frac{\partial y_1}{\partial x_2} = 2 \\ y_2 = 5x_1 + 4x_2 - 9 \Rightarrow \frac{\partial y_2}{\partial x_1} = 5 \wedge \frac{\partial y_2}{\partial x_2} = 4 \end{cases} \Rightarrow \frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}.$$

c) $$\begin{cases} x = 2r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = 2\cos\theta \wedge \frac{\partial x}{\partial \theta} = -2r\sin\theta \\ y = 3r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = 3\sin\theta \wedge \frac{\partial y}{\partial \theta} = 3r\cos\theta \end{cases} \Rightarrow$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \end{bmatrix} \quad (r>0,\ 0\le\theta\le 2\pi).$$

d) $$\begin{cases} y_1 = x_1^2 - x_2^2 \Rightarrow \frac{\partial y_1}{\partial x_1} = 2x_1 \wedge \frac{\partial y_1}{\partial x_2} = -2x_2 \\ y_2 = 2x_1x_2 \Rightarrow \frac{\partial y_2}{\partial x_1} = 2x_2 \wedge \frac{\partial y_2}{\partial x_2} = 2x_1 \end{cases} \Rightarrow f'(\mathbf{x}) = \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \end{bmatrix}.$$

Övning 3.9 (s. 62)

a) $$f'(0,0) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow df(0,0) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 2h_1 \\ h_1+h_2 \end{bmatrix}.$$

$$f'(2,-2) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow df(2,-2) = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 4h_1+2h_2 \\ h_1+h_2 \end{bmatrix}.$$

b) $$f'(1,2) = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \Rightarrow df(1,2) = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 3h_1+2h_2 \\ 5h_1+4h_2 \end{bmatrix}.$$

forts.

Matrisen $f'(x)$ är konstant, så vi har

$$f'(1,2)=f'(1,-1)\ ,\ df(1,2)=df(1,-1).$$

c) $f'(1,0)=\begin{bmatrix}2&0\\0&3\end{bmatrix} \Rightarrow df(1,0)=\begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix}=\begin{bmatrix}2h_1\\3h_2\end{bmatrix};$

$f'(2,\frac{\pi}{2})=\begin{bmatrix}0&-4\\3&0\end{bmatrix} \Rightarrow df(2,\frac{\pi}{2})=\begin{bmatrix}0&-4\\3&0\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix}=\begin{bmatrix}-4h_2\\3h_1\end{bmatrix}.$

d) $f'(1,1)=\begin{bmatrix}2&-2\\2&2\end{bmatrix} \Rightarrow df(1,1)=\begin{bmatrix}2&-2\\2&2\end{bmatrix}\begin{bmatrix}h_1\\h_2\end{bmatrix}=\begin{bmatrix}2h_1-2h_2\\2h_1+2h_2\end{bmatrix}.$

$f'(0,0)=0 \Rightarrow df(0,0)=0.$

Övning 3.10 (s. 62)

För att en avbildning f ska vara konform i punkten a, krävs det att $f'(a)$ är en multipel av en ON-matris. Detta är fallet i d) (dock inte i (0,0)).

$$f'(x)=\begin{bmatrix}2x_1&-2x_2\\2x_2&2x_1\end{bmatrix}=2|x|\cdot\begin{bmatrix}x_1/|x|&-x_2/|x|\\x_2/|x|&x_1/|x|\end{bmatrix}.$$

Läs Ex. 9 på s. 116 i grundboken.

Övning 3.11 (s. 62)

$f(x)=(x_1^2+x_2^2,\ x_1),\quad g(t)=(t_1+4t_2,\ -2t_1+2t_2).$

$$\begin{cases} y_1=x_1^2+x_2^2 \Rightarrow \frac{\partial y_1}{\partial x_1}=2x_1 \wedge \frac{\partial y_1}{\partial x_2}=2x_2 \\ y_2=x_1 \Rightarrow \frac{\partial y_2}{\partial x_1}=1 \wedge \frac{\partial y_2}{\partial x_2}=0 \end{cases} \Rightarrow f'(x)=\begin{bmatrix}2x_1&2x_2\\1&0\end{bmatrix}.$$

$$\begin{cases} x_1=t_1+4t_2 \\ x_2=-2t_1+2t_2 \end{cases} \Rightarrow g'(t)=\frac{\partial(x_1,x_2)}{\partial(t_1,t_2)}=\begin{bmatrix}1&4\\-2&2\end{bmatrix};$$

$$(f\circ g)'(t)=f'(g(t))\cdot g'(t)=\begin{bmatrix}2t_1+8t_2&-4t_1-4t_2\\1&0\end{bmatrix}\begin{bmatrix}1&4\\-2&2\end{bmatrix}=$$

$$=\begin{bmatrix}10t_1&40t_2\\1&4\end{bmatrix};\ \text{(enligt kedjeregeln).}$$

Annan metod

$$h(t)=f(g(t))=((t_1+4t_2)^2+(-2t_1+2t_2)^2,\ t_1+4t_2)=$$
$$=(5t_1^2+20t_2^2,\ t_1+4t_2)=(z_1,z_2);$$

$$\begin{cases} z_1=5t_1^2+20t_2^2 \\ z_2=t_1+4t_2 \end{cases} \Rightarrow h'(t)=\begin{bmatrix}\frac{\partial z_1}{\partial t_1}&\frac{\partial z_1}{\partial t_2}\\ \frac{\partial z_2}{\partial t_1}&\frac{\partial z_2}{\partial t_2}\end{bmatrix}=\begin{bmatrix}10t_1&40t_2\\1&4\end{bmatrix}.$$

Övning 3.12 (s. 62)

$$h(t)=f(g(t))=\left(\sqrt{\tfrac{x_1+x_2}{2}},\ \sqrt{\tfrac{x_1-x_2}{2}}\right)=(t_1,t_2),\ \text{ty}$$

$$g(t_1,t_2)=(t_1^2+t_2^2,\ t_1^2-t_2^2)=(x_1,x_2)\Leftrightarrow\begin{cases}x_1=t_1^2+t_2^2\\ x_1=t_1^2-t_2^2\end{cases}\Leftrightarrow$$

$$\Rightarrow\begin{cases}\frac{x_1+x_2}{2}=t_1^2\\ \frac{x_1-x_2}{2}=t_2^2\end{cases}\Leftrightarrow\begin{cases}\sqrt{\frac{x_1+x_2}{2}}=t_1\\ \sqrt{\frac{x_1-x_2}{2}}=t_2\end{cases},\ (t_1,t_2>0).$$

$$h'(t)=(f\circ g)'(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix}=E\quad(\underline{\text{enhetsmatrisen}})$$

Övning 3.13 (s. 62)

$$\begin{bmatrix}\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x}&\frac{\partial g}{\partial y}\end{bmatrix}=\begin{bmatrix}x-y&-x\\ y&x+y\end{bmatrix}\Leftrightarrow\begin{cases}\frac{\partial f}{\partial x}=x-y\\ \frac{\partial f}{\partial y}=-x\end{cases}(*)\ \wedge\begin{cases}\frac{\partial g}{\partial x}=y\\ \frac{\partial g}{\partial y}=x+y\end{cases}(**)$$

(i) $\frac{\partial f}{\partial x}=x-y\Leftrightarrow f(x,y)=\frac{1}{2}x^2-xy+\phi(y)\Rightarrow\frac{\partial f}{\partial y}=-x+\phi'(y)\overset{(*)}{=}-x\Leftrightarrow\phi'(y)=0\Leftrightarrow\phi(y)=C_1\Rightarrow\underline{f(x,y)=\frac{1}{2}x^2-xy+C_1}.$

(ii) $\frac{\partial g}{\partial x}=y\Leftrightarrow g(x,y)=xy+\psi(y)\Rightarrow\frac{\partial g}{\partial y}=x+\psi'(y)\overset{(**)}{=}x+y\Leftrightarrow\psi'(y)=y\Leftrightarrow\psi(y)=\frac{1}{2}y^2+C_2\Leftrightarrow\underline{g(x,y)=\frac{1}{2}y^2+xy+C_2}.$

Svar. Den givna matrisen är en funktionalmatris. För övrigt se ovan.

Funktionaldeterminanter

Övning 3.14 (s. 63)

a) $\underline{f(x_1,x_2)=(x_1^2+2x_2,\ x_1+x_2)}.$

$$J(x_1,x_2)=\det f'(x)=\begin{vmatrix}2x_1&2\\1&1\end{vmatrix}=2x_1-2=\underline{2(x_1-1)}.$$

b) $\underline{f(x_1,x_2)=(3x_1+2x_2-5,\ 5x_1+4x_2-9)}$

$$J(x_1,x_2)=\det f'(x)=\begin{vmatrix}3&2\\5&4\end{vmatrix}=12-10=\underline{2}.$$

c) $\underline{f(r,\theta)=(2r\cos\theta,\ 3r\sin\theta)}$

$$J(r,\theta)=\det f'(r,\theta)=\begin{vmatrix}2\cos\theta&-2r\sin\theta\\3\sin\theta&3r\cos\theta\end{vmatrix}=\underline{6r}.$$

d) $\underline{f(x_1,x_2)=(x_1^2-x_2^2,\ 2x_1x_2)}.$

$$J(x_1,x_2)=\det f'(x)=\begin{vmatrix}2x_1&-2x_2\\2x_2&2x_1\end{vmatrix}=\underline{4(x_1^2+x_2^2)}.$$

Övning 3.15 (s. 63)

$$J(t_1,t_2)=\det\begin{bmatrix}10t_1&40t_2\\1&4\end{bmatrix}=\underline{40(t_1-t_2)}.$$

Övning 3.16 (s. 63)

Determinanten av enhetsmatrisen är 1.

Övning 3.17 (s. 63)

Se nästa sida.

$$\begin{cases} u = x^2+y^2 \Rightarrow \frac{\partial u}{\partial x} = 2x \wedge \frac{\partial u}{\partial y} = 2y\,; \\ v = \sin(x^2+y^2) \Rightarrow \frac{\partial v}{\partial x} = 2x\cos|\mathbf{x}|^2 \wedge \frac{\partial v}{\partial y} = 2y\cos|\mathbf{x}|^2 . \end{cases}$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 2x & 2y \\ 2x\cos|\mathbf{x}|^2 & 2y\cos|\mathbf{x}|^2 \end{vmatrix} = \cos|\mathbf{x}|^2 \begin{vmatrix} 2x & 2y \\ 2x & 2y \end{vmatrix} = 0.$$

Övning 3.18 (s. 63)

$$\begin{cases} u = h(g(x,y)) \Rightarrow \frac{\partial u}{\partial x} = h'(g(\mathbf{x}))\frac{\partial g}{\partial x} \wedge \frac{\partial u}{\partial y} = h'(g(\mathbf{x}))\frac{\partial g}{\partial y}\,; \\ v = g(x,y) \Rightarrow \frac{\partial v}{\partial x} = \frac{\partial g}{\partial x} \wedge \frac{\partial v}{\partial y} = \frac{\partial g}{\partial y}. \end{cases}$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} h'(g(\mathbf{x}))\frac{\partial g}{\partial x} & h'(g(\mathbf{x}))\frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} =$$

$$= h'(g(\mathbf{x})) \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0.$$

Övning 3.19 (s. 63)

$$\begin{cases} u = x+ay \\ v = 2x+3y \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 1 & a \\ 2 & 3 \end{vmatrix} = 3-2a\,;$$

Avbildningen beskriver ett koordinatbyte om och endast om $\frac{d(u,v)}{d(x,y)} \neq 0$ om och endast om $3-2a \neq 0$ om och endast om $a \neq \frac{3}{2}$.

Övning 3.20 (s. 63)

a) $$\begin{cases} u = x+y+z \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 1 \\ v = x-y+z \Rightarrow \frac{\partial v}{\partial x} = 1 \wedge \frac{\partial v}{\partial y} = -1 \wedge \frac{\partial v}{\partial z} = 1 \\ w = x^2+y^2+z^2-2yz \Rightarrow \frac{\partial w}{\partial x} = 2x \wedge \frac{\partial w}{\partial y} = 2y-2z \wedge \frac{\partial w}{\partial z} = 2z-2y \end{cases}$$

$$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix} \equiv 0, \text{ ty}$$

kolonn 2 = −(kolonn 3).

b) Nej, ty funktionaldeterminanten är lika med noll oberoende av $\mathbf{x}$. Se även facit.

Övning 3.21 (s. 63)

a) $$\begin{cases} x+2y = u \quad (-2) \\ 3x+4y = v \end{cases} \Leftrightarrow \begin{cases} x+2y = u \\ x = -2u+v \end{cases} \Leftrightarrow \begin{cases} x = -2u+v \\ y = \frac{3}{2}u - \frac{1}{2}v \end{cases}$$

b) $u = x+2y \Rightarrow \frac{\partial u}{\partial x} = 1.$

c) Se under a).

d) $x = -2u+v \Rightarrow \frac{\partial x}{\partial u} = -2.$

Anm. Avbildningen är linjär och $\det \neq 0$

Implicita funktioner

Övning 3.22 (s. 64)

$x^3y+2y^3x=3,\ y=y(x)$.

$\frac{d}{dx}(x^3y+2y^3x)'=0 \Rightarrow 3x^2y+x^3y'+2y^3+6xy^2y'=0$

$(x,y)=(1,1) \Rightarrow 3+y'(1)+2+6y'(1)=0 \Leftrightarrow y'(1)=-\frac{5}{7}$.

Övning 3.23 (s. 64)

$x^y+\sin y=1,\ y=y(x)$ $\qquad (x^y=e^{y\ln x})$

$\frac{d}{dx}(x^y+\sin y)=0 \Rightarrow x^y\left(\frac{y}{x}+y'\ln x\right)+\cos y\, y'=0$

$\Leftrightarrow yx^{y-1}+y'x^y\ln x+\cos y\, y'=0 \Leftrightarrow yx^{y-1}=$

$=(-\cos y-x^y\ln x)y' \Leftrightarrow y'=-\frac{yx^{y-1}}{\cos y+x^y\ln x}$.

Övning 3.24 (s. 64)

$\sin xy-\ln(x+y)=0$ är nivåkurva till funktionen

$$f(x,y)=\sin xy-\ln(x+y)$$

$\frac{\partial f}{\partial y}=x\cos xy-\frac{1}{x+y} \Rightarrow f'_y(0,1)=-1\neq 0$.

Enligt implicita funktionssatsen (Sats 3) kan y framställas som funktion av x i en omgivning till punkten (0,1).

$\sin(xy)-\ln(x+y)=0 \Rightarrow \frac{d}{dx}\sin(xy)-\frac{d}{dx}\ln(x+y)=0 \Rightarrow$

$\Rightarrow (\cos(xy))\frac{d}{dx}(xy)-\frac{1}{x+y}\frac{d}{dx}(x+y)=0 \Leftrightarrow \cos xy(xy'+y)-$

$-\frac{1}{x+y}(1+y')=0 \Leftrightarrow (x+y)(xy'+y)\cos(xy)=1+y';\ (*)$

$(x,y)=(0,1) \overset{(*)}{\Rightarrow} 1=1+y'(0) \Leftrightarrow y'(0)=0$.

Övning 3.25 (s. 64)

$x^5+y^3+z^4-(x^2+y^2)z=1$ är en nivåyta till

$$f(x,y,z)=x^5+y^3+z^4-(x^2+y^2)z.$$

$\frac{\partial f}{\partial z}=4z^3-x^2-y^2 \Rightarrow f'_z(1,1,1)=2\neq 0$.

Enligt implicita funktionssatsen går det att i en omgivning till punkten (1,1,1) lösa ut z som funktion av (x,y) i ekvationen $f(x,y,z)=1$.

$x^5+y^3+z^4-(x^2+y^2)z=1 \overset{\partial_x}{\underset{\partial_y}{\Rightarrow}} \begin{cases} 5x^4+4z^3\frac{\partial z}{\partial x}-2xz-(x^2+y^2)\frac{\partial z}{\partial x}=0 \\ 3y^2+4z^3\frac{\partial z}{\partial y}-2yz-(x^2+y^2)\frac{\partial z}{\partial y}=0 \end{cases};$

$(x,y,z)=(1,1,1) \Rightarrow \begin{cases} 5+4z'_x(1,1)-2-2z'_x(1,1)=0 \\ 3+4z'_y(1,1)-2-2z'_y(1,1)=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} 2z'_x(1,1)=-3 \\ 2z'_y(1,1)=-1 \end{cases} \Leftrightarrow \begin{cases} z'_x(1,1)=-\frac{3}{2} \\ z'_y(1,1)=-\frac{1}{2} \end{cases}.$

Övning 3.26 (s.64)

Den givna ekvationen är en nivåyta till

$$f(x,y,z)=x^3+y^3+z^3+x^2z-yz-z.$$

$\frac{\partial f}{\partial z}=3z^2+x^2-y-1 \Rightarrow f'_z(0,0,0)=-1\neq 0.$

Det går således att framställa z som funktion av (x,y) i en omgivning till (0,0,0) (Sats 3).

$$x^3+y^3+z^3+x^2z-yz-z=0 \Rightarrow \begin{cases} 3x^2+3z^2\frac{\partial z}{\partial x}+x^2\frac{\partial z}{\partial x}+2xz-(y+1)\frac{\partial z}{\partial x}=0 \\ 3y^2+3z^2\frac{\partial z}{\partial y}+x^2\frac{\partial z}{\partial y}-z-(y+1)\frac{\partial z}{\partial y}=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (3z^2+x^2-y-1)\frac{\partial z}{\partial x}=-3x^2-2xz \\ (3z^2+x^2-y-1)\frac{\partial z}{\partial y}=-3y^2+z \end{cases} \Leftrightarrow \begin{cases} \frac{\partial z}{\partial x}=-\frac{3x^2+2xz}{3z^2+x^2-y-1} \\ \frac{\partial z}{\partial y}=\frac{z-3y^2}{3z^2+x^2-y-1} \end{cases}.$$

Övning 3.27 (s. 64)

a) $(x,y,z)=(0,1,1) \Rightarrow VL=1+1+0-2=0=HL.$

Den givna ekvationen är nivåyta till

$$f(x,y,z)=e^{z-1}+zy+x-2y^3.$$

$\operatorname{grad} f(x,y,z)=(1,z-6y^2,e^{z-1}+y) \Rightarrow \operatorname{grad} f(0,1,1)=(1,-5,2).$

Planets ekvation ges alltså av

$\operatorname{grad} f(0,1,1)\cdot(x,y-1,z-1)=0 \Leftrightarrow \underline{x-5y+2z=-3}.$

b) $\frac{\partial f}{\partial z}=e^{z-1}+y \Rightarrow f'_z(0,1,1)=2\neq 0.$

Enligt implicita funktionssatsen går det att framställa z som funktion av (x,y) i den föreslagna punktens omedelbara omgivning.

Anm. I en liten omgivning till (0,1,1) kan man använda den linjära approximationen i a), dvs. $f(x,y,z)\approx x-5y+2z$. Denna koefficient för z är $\neq 0$, så tangentplanet är inte vertikalt.

Övning 3.28 (s.64)

a) $x^2y^3-3xy^2-9y+9=0 \overset{d_x}{\Rightarrow} 2xy^3+3x^2y^2y'-3y^2-6xyy'-$

$-9y'=0 \Leftrightarrow (3x^2y^2-6xy-9)y'=3y^2-2xy^3$; (*)

$(x,y)=(0,1) \overset{(*)}{\Rightarrow} -9y'=3 \Leftrightarrow y'=-\frac{1}{3}=k_t.$

Enpunktsformeln ger $y=-\frac{1}{3}x+1.$

b) Den givna ekvationen är en nivåkurva till

$$f(x,y)=x^2y^3-3xy^2-9y+9$$

$\frac{\partial f}{\partial y}=3x^2y^2-6xy-9=3(xy-3)(xy+1).$

Enligt implicita funktionssatser går det bra när $f'_y \neq 0$.

$f'_y(x,y)=0 \Rightarrow xy-3=0 \vee xy+1=0 \Leftrightarrow y=\frac{3}{x} \vee y=-\frac{1}{x}$.

$f(x,\frac{3}{x})=0 \Rightarrow -\frac{27}{x}+9=0 \wedge y=\frac{3}{x} \Leftrightarrow (x,y)=(3,1)$.

$f(x,-\frac{1}{x})=0 \Rightarrow \frac{5}{x}+9=0 \wedge y=-\frac{1}{x} \Leftrightarrow (x,y)=(-\frac{5}{9},\frac{9}{5})$.

Svar: a) $x+3y-3=0$; b) I alla (x,y) utom $(3,1)$ och $(-\frac{5}{9},\frac{9}{5})$. F.ö. se ovan.

Övning 2.39 (s. 64)

a) Den givna ekvationen är nivåkurva till

$$f(x,y)=e^y-x\cos y-1.$$

$\frac{\partial f}{\partial y}=e^y+x\sin y \Rightarrow f'_y(0,0)=1\neq 0.$

Nivåkurvan kan således framställas som funktion av x i en omgivning till $(0,0)$.

$e^y-x\cos y-1=0 \Rightarrow \frac{d}{dx}(e^y-x\cos y)=0 \Rightarrow e^y\cdot y'-\cos y+x\sin y\cdot y'=0 \Leftrightarrow (e^y+x\sin y)y'=\cos y$; (*)

$(x,y)=(0,0) \overset{(*)}{\Rightarrow} y'(0)=1=k_t \Rightarrow y=x$ (tangenten).

b) Vi behöver andraderivatan, så vi deriverar (*) en gång till implicit.

forts.

$(e^y+x\sin y)y''=-(\sin y+e^y+\sin y+x\cos y)y'$;

$(x,y)=(0,0) \Rightarrow y''(0)=y'(0)=-1 \Rightarrow a_2=\frac{y''(0)}{2!}=-\frac{1}{2}$.

Resultat: a) $y=x$, b) $y=x-\frac{1}{2}x^2+O(x^3)$.

Övning 3.30 (s. 65)

a) $y^3-3y=x$ är en nivåkurva till

$$f(x,y)=y^3-3y-x.$$

$\frac{\partial f}{\partial y}=3y^2-3 \Rightarrow f'_y(0,0)=-3\neq 0$;

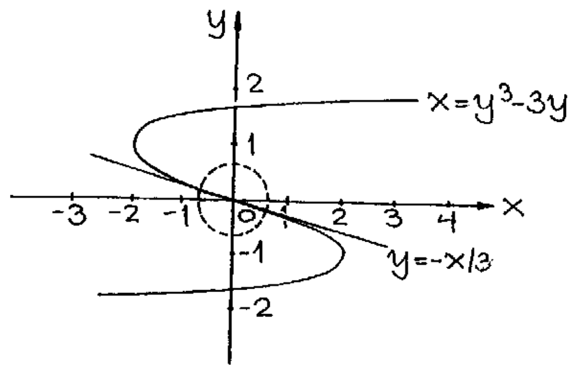
Enligt implicita funktionssatsen går det bra.

b) $y^3-3y=x \Rightarrow \frac{d}{dx}(y^3-3y)=1 \Rightarrow (3y^2-3)y'=1$ (*) $\Rightarrow$

$\Rightarrow \frac{d}{dx}3(y^2-1)y'=0 \Rightarrow (y^2-1)y''=2yy'^2$; (**)

$(x,y)=(0,0) \overset{(*)}{\Rightarrow} y'=-\frac{1}{3} \overset{(**)}{\Rightarrow} y''=0$;

Svar: a) Se ovan!, b) $y=-\frac{1}{3}x+O(x^3)$.



Övning 3.31 (s. 65)

$x^2+z^2-z^2=2$ är nivåyta till $f(x,y,z)=x^2+y^2-z^2$.

$x+y=2e^z$ är nivåyta till $g(x,y,z)=x+y-2e^z$.

$$\frac{d(f,g)}{d(y,z)}=\begin{vmatrix}\frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\end{vmatrix}=\begin{vmatrix}2y & -2z\\ 1 & -2e^z\end{vmatrix}=2z-4ye^z \Rightarrow J(1,0)\neq 0.$$

$x=t \Rightarrow \mathbf{x}(t)=(t,y(t),z(t))$.

$$\begin{cases} f(t,y(t),z(t))=2 \Rightarrow t^2+y^2(t)-z(t)^2=2\\ g(t,y(t),z(t))=0 \Rightarrow t+y(t)=2e^{z(t)}\end{cases} \text{(derivera)} \Rightarrow$$

$$\Rightarrow \begin{cases} 2t+2y(t)y'(t)-2z(t)z'(t)=0\\ 1+y'(t)=2e^{z(t)}z'(t)\end{cases} ((x,y,z)=(1,1,0)) \Rightarrow$$

$$\Rightarrow \begin{cases} 2+2y'(1)=0\\ 1+y'(1)=2z'(1)\end{cases} \Leftrightarrow \begin{cases} y'(1)=-1\\ z'(1)=0\end{cases} \Rightarrow \mathbf{x}'(1)=(1,-1,0).$$

Övning 3.32 (s. 65)

$x^2-y^2-z^2=-1$ är nivåyta till $f(x,y,z)=x^2-y^2-z^2$.

$x^2+2y^2+3z^2=6$ är nivåyta till $g(x,y,z)=x^2+2y^2+3z^2$.

$$\frac{d(f,g)}{d(x,y)}=\begin{vmatrix}2x & -2y\\ 2x & 4y\end{vmatrix}=8xy+4xy=12xy \Rightarrow J(1,1)=12\neq 0.$$

forts.

$\mathbf{x}(t)=(x(t),y(t),t) \Rightarrow \mathbf{x}'(t)=(x'(t),y'(t),1)$.

$$\begin{cases} f(\mathbf{x}(t))=x^2(t)-y(t)^2-t^2=-1\\ g(\mathbf{x}(t))=x^2(t)+2y(t)^2+3t^2=6\end{cases} \text{(derivera m.a.p. } t) \Rightarrow$$

$$\Rightarrow \begin{cases} 2x(t)x'(t)-2y(t)y'(t)-2t=0\\ 2x(t)x'(t)+4y(t)y'(t)+6t=0\end{cases} \Rightarrow \begin{cases} x'(1)-y'(1)=1\\ x'(1)+2y'(1)=-3\end{cases} \Rightarrow$$

$$\Rightarrow x'(1)=-\frac{1}{3} \wedge y'(1)=-\frac{4}{3} \Rightarrow \mathbf{x}'(1)=\left(-\frac{1}{3},-\frac{4}{3},1\right)=-\frac{1}{3}(1,4,-3).$$

Resultat: Tangentens ekvation är $\mathbf{x}(t)=(1,1,1)+t(1,4,-3)$.

Blandade problem

Övning 3.33 (s. 65)

$$\begin{cases} y_1=f(x_1)\cos\alpha x_2\\ y_2=f(x_1)\sin\alpha x_2\\ y_3=x_3\end{cases} \Rightarrow \frac{dy}{dx}=\begin{vmatrix} f'(x_1)\cos\alpha x_2 & -\alpha f(x_1)\sin\alpha x_2 & 0\\ f'(x_1)\sin\alpha x_2 & \alpha f(x_1)\cos\alpha x_2 & 0\\ 0 & 0 & 1\end{vmatrix}$$

$=\alpha f(x_1)f'(x_1)\cos^2(\alpha x)+\alpha f(x_1)f'(x_1)\sin^2(\alpha x)=\alpha f(x_1)f'(x_1)$;

$$\frac{d(y_1,y_2,y_3)}{d(x_1,x_2,x_3)}=1 \Rightarrow \alpha f(x_1)f'(x_1)=1 \Leftrightarrow \alpha f(x_1)^2=2x_1+C$$

$$\Leftrightarrow f(x_1)^2=\frac{2x_1+C}{\alpha} \Leftrightarrow f(x_1)=\pm\sqrt{\frac{2x_1+C}{\alpha}}.$$

Övning 3.34 (s. 65)

$x^3-3ax+2=0 \Rightarrow x(0)^3-3\cdot 0\cdot x(0)+2=0 \Leftrightarrow x(0)=-\sqrt[3]{2}$

$\frac{d}{da}(x^3-3ax+2)=0 \Rightarrow 3x^2x'-3x-3ax'=0 \Rightarrow (a=0) \Rightarrow$

$$\Rightarrow 3\cdot x(0)^2\cdot x'(0)-3x(0)=0 \Leftrightarrow x'(0)=\frac{1}{x(0)}=-\frac{1}{\sqrt{2}}.$$

Övning 3.35 (s.66)

$$F'_x(a,b,c)\neq 0 \Rightarrow x=f(y,z)\Rightarrow dx=\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz;\quad (1)$$

$$F'_y(a,b,c)\neq 0 \Rightarrow y=g(x,z)\Rightarrow dy=\left(\frac{\partial y}{\partial x}\right)_z dx+\left(\frac{\partial y}{\partial z}\right)_x dz;\quad (2)$$

$$F'_z(a,b,c)\neq 0 \Rightarrow z=h(x,y)\Rightarrow dz=\left(\frac{\partial z}{\partial x}\right)_y dx+\left(\frac{\partial z}{\partial y}\right)_x dy;\quad (3)$$

$$(1)\Rightarrow dx=\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz=(2)=$$

$$=\left(\frac{\partial x}{\partial y}\right)_z\left\{\left(\frac{\partial y}{\partial x}\right)_z dx+\left(\frac{\partial y}{\partial z}\right)_x dz\right\}+\left(\frac{\partial x}{\partial z}\right)_y dz=$$

$$=\left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial x}\right)_z dx+\left\{\left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial z}\right)_x+\left(\frac{\partial x}{\partial z}\right)_y\right\}dz;\quad (4)$$

$$(3)\Rightarrow dz=\left(\frac{\partial z}{\partial x}\right)_y dx+\left(\frac{\partial z}{\partial y}\right)_x dy=(1)=$$

$$=\left(\frac{\partial z}{\partial x}\right)_y\left\{\left(\frac{\partial x}{\partial y}\right)_z dy+\left(\frac{\partial x}{\partial z}\right)_y dz\right\}+\left(\frac{\partial z}{\partial y}\right)_x dy=$$

$$=\left(\frac{\partial z}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial z}\right)_y dz+\left\{\left(\frac{\partial z}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial y}\right)_z+\left(\frac{\partial z}{\partial y}\right)_x\right\}dy\quad (5).$$

Vi sätter $dx=0$ i (4) (dvs. håller x = konstant):

$$\left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial z}\right)_x+\left(\frac{\partial x}{\partial z}\right)_y=0 \Leftrightarrow \left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial z}\right)_x=-\left(\frac{\partial x}{\partial z}\right)_y;\quad (6)$$

Vi sätter $dy=0$ i (5) och får

$$dz=\left(\frac{\partial z}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial z}\right)_y dz \Leftrightarrow \left(\frac{\partial z}{\partial x}\right)_y=\left(\frac{\partial x}{\partial z}\right)_y^{-1}\quad (7)$$

Sätter vi (7) i (6), alternativt multiplicerar vi (6) med (7), så får vi slutligen relationen

$$\left(\frac{\partial x}{\partial y}\right)_z\cdot\left(\frac{\partial y}{\partial z}\right)_x\cdot\left(\frac{\partial z}{\partial x}\right)_y=-1.$$

Detta är Barkhausens rörformel.

Övning 3.36 (s.66)

$$u=x^2-y^2,\ v=2xy;\quad \frac{d(u,v)}{d(x,y)}=4(x^2+y^2)\Rightarrow J(1,1)\neq 0.$$

Avbildningen är bijektiv, dvs. inverterbar i (1,1).

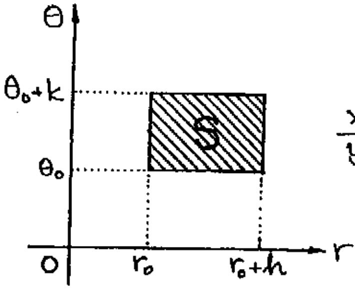
$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{bmatrix}2x & -2y\\ 2y & 2x\end{bmatrix}\Rightarrow \frac{\partial(u,v)}{\partial(x,y)}\Big|_{(1,1)}=\begin{bmatrix}2 & -2\\ 2 & 2\end{bmatrix}\Rightarrow \frac{\partial(x,y)}{\partial(u,v)}\Big|_{(0,2)}=$$

$$=\left(\frac{\partial(u,v)}{\partial(x,y)}\Big|_{(1,1)}\right)^{-1}=\frac{1}{8}\begin{bmatrix}2 & 2\\ -2 & 2\end{bmatrix}\Rightarrow \begin{cases}\frac{\partial u}{\partial x}\Big|_{(1,1)}=2\\ \frac{\partial x}{\partial u}\Big|_{(0,2)}=\frac{1}{4}.\end{cases}$$

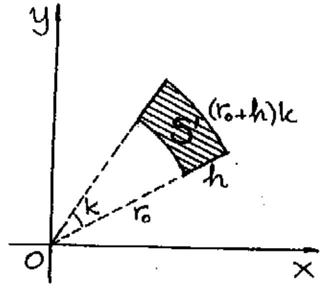
Övning 3.37 (s.66)

a) $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\Rightarrow J(r,\theta)=\frac{d(x,y)}{d(r,\theta)}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r.$

b)



$\dfrac{x=r\cos\theta}{y=r\sin\theta}$



$$\frac{\mu(S')}{\mu(S)} = \frac{\frac{1}{2}[(r_0+h)^2-r_0^2]\cdot k}{h\cdot k} = r_0 + \frac{1}{2}h \; ; \; (\Delta r = h, \Delta\theta = k).$$

$$\lim_{(h,k)\to(0,0)} \frac{\mu(S')}{\mu(S)} = \lim_{h\to 0}\left(r_0 + \frac{1}{2}h\right) = r_0 = \frac{d(x,y)}{d(r,\theta)}\Big|_{(r_0,\theta_0)}$$

Övning 3.38 (s.64)

a) $f(x,y) = 2\cos(xy) - x \Rightarrow f(1,\frac{\pi}{3}) = 0 \Leftrightarrow 2\cos xy - x = 0.$

$\frac{\partial f}{\partial y} = -2x\sin(xy) \Rightarrow f'_y(1,\frac{\pi}{3}) = -\sqrt{2}$;

Enligt implicita funktionssatsen kan man uttrycka y som funktion av x.

$2\cos xy - x = 0 \Rightarrow -2\sin xy \cdot (xy'+y) - 1 = 0 \Leftrightarrow$

$\Leftrightarrow xy' + y = -\frac{1}{2\sin xy} \Leftrightarrow xy' = -y - \frac{1}{2\sin xy} \Leftrightarrow$

$\Leftrightarrow y' = -\frac{2y\sin xy + 1}{2x\sin xy}$

b) $2\cos xy - x = 0 \Leftrightarrow \cos xy = \frac{x}{2} \Leftrightarrow xy = \arccos\frac{x}{2}, \; |x| \leq 2 \Leftrightarrow$

$\Leftrightarrow y = \frac{1}{x}\arccos\frac{x}{2}, \; |x| \leq 2, \; x \neq 0.$

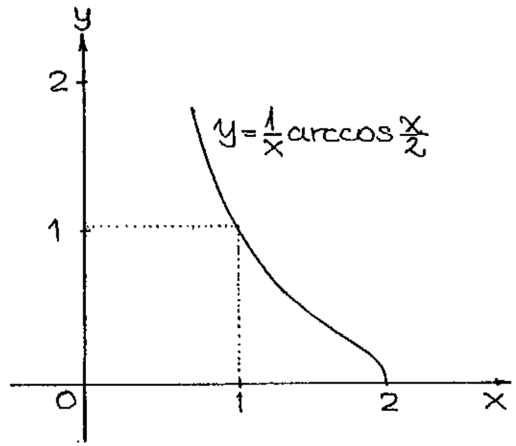
c) $y' = -\frac{1}{x^2}\arccos\frac{x}{2} - \frac{1}{2x}\frac{1}{\sqrt{1-x^2/4}}$

d) $(x,y) = (1,\frac{\pi}{3}) \Rightarrow y' = -\frac{2y\sin xy + 1}{2x\sin xy} = -\frac{1}{3}(\pi + \sqrt{3}).$

$y'(1) = -\arccos\frac{1}{2} - \frac{1}{2}\frac{1}{\sqrt{3/4}} = -\frac{\pi}{3} - \frac{1}{\sqrt{3}} = -\frac{1}{3}(\pi + \sqrt{3}).$

e)

x	0,7	0,8	0,9	1,1	1,2	1,3
y	1,733	1,449	1,227	0,899	0,773	0,664



Övning 3.39 (s.67)

Den givna ekvationen är en nivåkurva till

$$f(x,y) = x^5 + xy + 1.$$

$\frac{\partial f}{\partial x} = 5x^4 + y \Rightarrow f'_x(-1,0) = +5 \neq 0.$

Enligt inversa funktionssatsen går det att framställa x som funktion av y i närheten av punkten $(-1,0)$

$x(0)^5 + x(0)\cdot 0 + 1 = 0 \Leftrightarrow \underline{x(0) = -1.}$

$x^5 + xy + 1 = 0 \Rightarrow \frac{d}{dy}(x^5 + xy + 1) = 0 \Rightarrow 5x^4\cdot x' + x'y + x = 0$

$\Leftrightarrow (5x^4+y)x' = -x \Leftrightarrow x' = -\frac{x}{5x^4+y} \Rightarrow \underline{x'(0) = \frac{1}{5}};$

$(5x^4+y)x' = -x \overset{\partial_y}{\Rightarrow} (5x^4+y)x'' = -(20x^3x' + 1 + 1)x' \Rightarrow$

$\Rightarrow \underline{x''(0) = \frac{2}{25}} \Rightarrow x = -1 + \frac{1}{5}y + \frac{1}{25}y^2 \Rightarrow x(0,1) = -0,9796.$

Övning 3.40 (s. 67)

$$z^3+z(y^2+1)+x^3-3x+y^2-8=0 \overset{\partial x}{\underset{\partial y}{\Rightarrow}} \begin{cases}(3z^2+y^2+1)\frac{\partial z}{\partial x}=3-3x^2\\ (3z^2+y^2+1)\frac{\partial z}{\partial y}=-2y(z+1)\end{cases}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0 \Rightarrow \begin{cases}3-3x^2=0\\ y=0\end{cases} \Leftrightarrow \underline{(x,y)=(1,0)} \vee \underline{(x,y)=(-1,0)}.$$

$$(x,y)=(1,0) \Rightarrow z^3+z+1-3-8=0 \Leftrightarrow z^3+z-10=0 \Leftrightarrow z=2.$$

$$(x,y)=(-1,0) \Rightarrow z^3+z-1+3-8=0 \Leftrightarrow z^3+z-6=0 \Leftrightarrow z=1,65.$$

$$(3z^2+y^2+1)\frac{\partial z}{\partial x}=3-3x^2 \overset{\partial x}{\Rightarrow} 6z\left(\frac{\partial z}{\partial x}\right)^2+(3z^2+y^2+1)\frac{\partial^2 z}{\partial x^2}=-6x$$

$$(3z^2+y^2+1)\frac{\partial z}{\partial y}=-2y(z+1) \overset{\partial x}{\underset{\partial y}{\Rightarrow}} \begin{cases}6z\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+(3z^2+y^2+1)\frac{\partial^2 z}{\partial x\partial y}=0\\ 6z\left(\frac{\partial z}{\partial y}\right)^2+(3z^2+y^2+1)\frac{\partial^2 z}{\partial y^2}=-2(z+1)\end{cases}$$

(i) $(x,y)=(1,0) \Rightarrow z''_{xx}=-\frac{6}{13} \wedge z''_{xy}=0 \wedge z''_{yy}=-\frac{6}{13} \Rightarrow$

$\Rightarrow Q(h,k)=-\frac{6}{13}(h^2+k^2)$ neg. definit $\Rightarrow$ maximum.

(ii) $(x,y)=(-1,0) \Rightarrow z''_{xx}>0 \wedge z''_{xy}=0 \wedge z''_{yy}<0 \Rightarrow$

$\Rightarrow Q(h,k)$ indefinit $\Rightarrow (0,-1)$ sadelpunkt

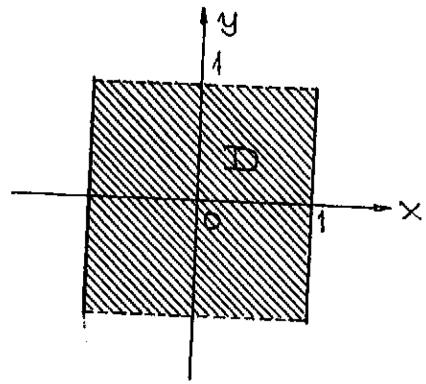
Resultat: Den enda extrempunkten är $(1,0)$; den är en lokal maximipunkt.

Optimering

Optimering över kompakta områden

Övning 4.1 (s. 78)

a) $f: D \to \mathbb{R}$; $f \in C^0$; $D=\{(x,y): |x|\leq 1, |y|<1\}$.



D är inte kompakt, så det är inte säkert att f antar sina extremvärden i D; dessa kan t.ex. ligga på den del av randen som fattas.

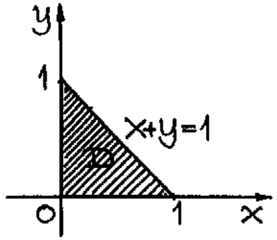
b) $f: D \to \mathbb{R}$, $f \in C^0$; $D=\{(x,y): x^2+y^2\leq 1\}$.

f är kontinuerlig på kompakt D, så den antar båda extremvärdena. (Sats 1.4, s. 33).

c) $f: D \to \mathbb{R}$, $f \in C^0$, $D=\{(x,y): x^2+y^2\geq 1\}$.

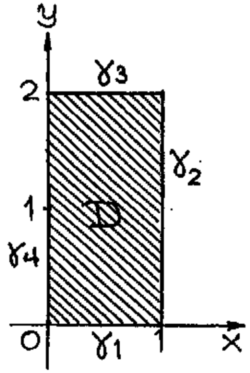
D är sluten men inte begränsad; den är således inte kompakt. Det är därför inte säkert att f antar sina extremvärden.

e) $f: D \to \mathbb{R}$, $f \in C^0$; $D = \{(x,y): x \geq 0, y \geq 0, x+y \leq 1\}$.



D är kompakt, så f antar sina extrema i D.

Övning 4.2 (s.78)



$\partial D = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

$D = [0,1] \times [0,2]$ (Se s. 362 i grundboken).

Anm. Normalt skriver man $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ men summan är ändå vanligast i topologin; γ_i måste då ha en riktning. En sådan behövs inte i fortsättningen.

$f \in C^0(D) = f$ är kontinuerlig på D; D är kompakt så både största och minsta värdet till f antas i D.

(i) $\mathring{D} = \{(x,y): 0 < x < 1, 0 < y < 2\} =]0,1[\times]0,2[$.

$\forall x \in \mathring{D}$: $f(x) = ye^x - xy^2 \Rightarrow \frac{\partial f}{\partial x} = ye^x - y^2 \wedge \frac{\partial f}{\partial y} = e^x - 2xy$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} y(e^x - y) = 0 \\ e^x - 2xy = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ e^x - 2xy = 0 \end{cases} \vee \begin{cases} y = e^x \\ e^x - 2xy = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = e^x \\ e^x - 2xe^x = 0 \end{cases} \Leftrightarrow \begin{cases} y = e^x \\ e^x(1-2x) = 0 \end{cases} \Leftrightarrow \begin{cases} 1-2x = 0 \\ y = e^x \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = e^{1/2} \end{cases}$$

$f(\frac{1}{2}, e^{1/2}) = e/2$

(ii) $\gamma_1 = [0,1] \times \{0\} = \{(x,0): 0 \leq x \leq 1\}$.

$f(x,0) \equiv 0$;

$\gamma_2 = \{1\} \times [0,2] = \{(1,y): 0 \leq y \leq 2\}$.

$f(1,y) = ey - y^2 = \phi(y)$, $0 \leq y \leq 2$

$\phi'(y) = e - 2y = 0 \Leftrightarrow y = \frac{e}{2} < 2$;

$\phi(0) = 0$, $\phi(\frac{e}{2}) = \frac{e^2}{4}$, $\phi(2) = 2e - 4 \approx 1{,}437$.

$\gamma_3 = [0,1] \times \{2\} = \{(x,2): 0 \leq x \leq 1\}$.

$f(x,2) = 2e^x - 4x = \psi(x)$, $0 \leq x \leq 1$

forts.

$\psi'(x)=2e^x-4=0 \Leftrightarrow e^x=2 \Leftrightarrow x=\ln 2$;

$\psi(0)=2$, $\psi(\ln 2)=4(1-\ln 2)\approx 1{,}227$, $\psi(1)=2e-4$.

$\gamma_4=\{0\}\times[0,2]=\{(0,y):\ 0\le y\le 2\}$.

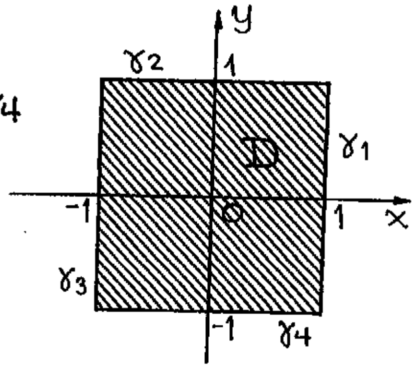
$f(0,y)=y$ växande; $f(0,0)=0$, $f(0,2)=2$.

Resultat: $\{\frac{e}{2}, 0, \frac{e^2}{4}, 2e-4, 2, 4(1-\ln 2)\}_{min}^{max}=\begin{cases}2\\0\end{cases}$.

Övning 4.2 (s.78)

$f(x,y)=xy+x^2y^2$, $D=[-1,1]\times[-1,1]$.

$\partial D=\gamma_1+\gamma_2+\gamma_3+\gamma_4$



(i) $\mathring{D}=]-1,1[\times]-1,1[=]-1,1[^2=\{(x,y):\ -1<x,y<1\}$.

$\forall \mathbf{x}\in\mathring{D}$: $\frac{\partial f}{\partial x}=y+2xy^2$, $\frac{\partial f}{\partial y}=x+2x^2y$.

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Leftrightarrow \begin{cases} y(1+2xy)=0\\ x(1+2xy)=0\end{cases} \overset{?}{\Leftrightarrow} \begin{cases}x=0\\ y=0\end{cases};\ f(0,0)=0.$$

Anm. Även alla punkter på hyperbeln

$y=-\frac{1}{2x}$ är kritiska; $f(x,-\frac{1}{2x})=-\frac{1}{2}+\frac{1}{4}=-\frac{1}{4}$.

(ii) $\gamma_1=\{1\}\times[-1,1]=\{(1,y):\ -1\le y\le 1\}$.

$f(1,y)=y+y^2=\phi(y)$, $-1\le y\le 1$.

$\phi'(y)=1+2y=0 \Leftrightarrow y=-\frac{1}{2}$;

$\phi(-1)=0$, $\phi(-\frac{1}{2})=-\frac{1}{4}$, $\phi(1)=2$.

$\gamma_2=[-1,1]\times\{1\}=\{(x,1):\ -1\le x\le 1\}$.

$f(x,1)=x+x^2=\psi(x)$, $-1\le x\le 1$.

$\psi'(x)=1+2x=0 \Leftrightarrow x=-\frac{1}{2}$;

$\psi(-1)=0$, $\psi(-\frac{1}{2})=-\frac{1}{4}$, $\psi(1)=2$.

$\gamma_3=\{-1\}\times[-1,1]=\{(-1,y):\ -1\le y\le 1\}$.

$f(-1,y)=y^2-y=\chi(y)$, $-1\le y\le 1$.

$\chi'(y)=2y-1=0 \Leftrightarrow y=\frac{1}{2}$;

$\chi(-1)=2$, $\chi(-\frac{1}{2})=\frac{3}{4}$, $\chi(1)=0$.

$\gamma_4=[-1,1]\times\{-1\}=\{(x,-1):\ -1\le x\le 1\}$.

$f(x,-1)=x^2-x=\omega(x)$, $-1\le x\le 1$.

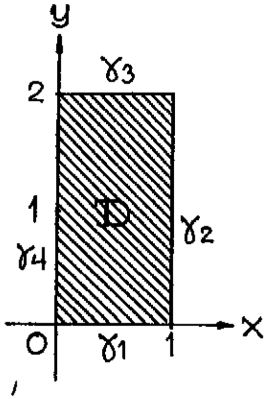
$\omega'(x)=2x-1=0 \Leftrightarrow \omega=\frac{1}{2}$.

$\omega(-1)=2$, $\omega(-\frac{1}{2})=\frac{3}{4}$, $\omega(1)=0$.

Resultat: $\{0, -\frac{1}{4}, 2, \frac{3}{4}\}_{min}^{max}=\begin{cases}2\\ -1/4\end{cases}$.

Övning 4.4 (s. 78)

$f(x,y) = xy^2 - x^2 - y, \quad D = [0,1] \times [0,2].$



(i) $\mathring{D} =]0,1[\times]0,2[= \{(x,y) : 0<x<1,\ 0<y<2\}.$

$\forall \mathbf{x} \in \mathring{D}: \ \frac{\partial f}{\partial x} = y^2 - 2x, \ \frac{\partial f}{\partial y} = 2xy - 1;$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} y^2 - 2x = 0 \\ 2xy - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y^3 = 1 \\ y^2 = 2x \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2} \\ y = 1 \end{cases};$$

$f(\frac{1}{2}, 1) = -3/4.$

(ii) $\gamma_1 = [0,1] \times \{0\} = \{(x,0) : 0 \le x \le 1\}.$

$f(x,0) = -x^2$ (avtagande).

$f(0,0) = 0, \ f(1,0) = -1.$

$\gamma_2 = \{1\} \times [0,2] = \{(1,y) : 0 \le y \le 2\}.$

$f(1,y) = y^2 - y - 1 = \phi(y), \ 0 \le y \le 2.$

$\phi'(y) = 2y - 1 = 0 \Leftrightarrow y = 1/2$

forts.

$\phi(0) = -1, \ \phi(\frac{1}{2}) = -\frac{5}{4}, \ \phi(2) = 1.$

$\gamma_3 = [0,1] \times \{2\} = \{(x,2) : 0 \le x \le 1\}.$

$f(x,2) = -x^2 + 4x - 2 = \psi(x), \ 0 \le x \le 1.$

$\psi'(x) = -2x + 4 > 0 \Rightarrow \psi$ strängt växande.

$\psi(0) = -2, \ \psi(1) = 1.$ (endast ändpunkterna).

$\gamma_4 = \{0\} \times [0,2] = \{(0,y) : 0 \le y \le 2\}.$

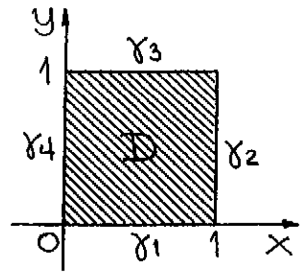
$f(0,y) = -y$ strängt avtagande.

$f(0,0) = 0, \ f(0,2) = -2.$

Resultat: $\{-3/4, 0, -1, -5/4, 1, -2\}^{max}_{min} = \begin{cases} 1 \\ -2 \end{cases}.$

Övning 4.5 (s. 78)

$f(x,y) = \frac{7y^2}{2(1+2x+3y)}, \quad D = [0,1]^2 = \{(x,y) : 0 \le x, y \le 1\}.$



(i) $\mathring{D} =]0,1[^2 = \{(x,y) : 0 < x, y < 1\}.$

$\forall \mathbf{x} \in \mathring{D}: \ \frac{\partial f}{\partial x} = -\frac{7y^2}{(1+2x+3y)^2}, \ \frac{\partial f}{\partial y} = \frac{7}{2} \frac{2y + 4xy + 3y^2}{(1+2x+3y)^2};$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases}7y^2=0\\ 7y(1+2x+3y)=0\end{cases} \Leftrightarrow \begin{cases} x=-\frac{1}{2} \\ y=0\end{cases} \Rightarrow \left(-\tfrac{1}{2},0\right)\notin \mathring{D}.$$

Inga kritiska punkter i det inre av D.

(ii) $\gamma_1=[0,1]\times\{0\}=\{(x,0):\ 0\le x\le 1\}$.

$f(x,0)\equiv 0$.

$\gamma_2=\{1\}\times[0,1]=\{(1,y):\ 0\le y\le 1\}$.

$f(1,y)=\frac{7}{6}\,\frac{y^2}{y+1}=\phi(y)$, $0\le y\le 1$;

$\phi'(y)=\frac{7}{6}\,\frac{2y+y^2}{(1+y)^2}>0 \Rightarrow \phi$ strängt växande.

$\phi(0)=0$, $\phi(1)=7/12$.

$\gamma_3=[0,1]\times\{1\}=\{(x,1):\ 0\le x\le 1\}$.

$f(x,1)=\frac{7}{4(x+2)}=\psi(x)$, $0\le x\le 1$

$\psi'(x)=-\frac{7}{4(x+2)^2}<0 \Rightarrow \psi$ strängt avtagande.

$\psi(0)=7/8$, $\psi(1)=7/12$.

$\gamma_4=\{0\}\times[0,1]=\{(0,y):\ 0\le y\le 1\}$.

$f(0,y)=\frac{7}{2}\,\frac{y^2}{3y+1}=\chi(y)$, $0\le y\le 1$.

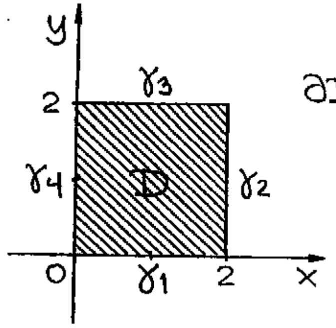
$\chi'(y)=-\frac{2y+3y^2}{(1+3y)^2}<0 \Rightarrow \chi$ strängt avtagande.

$\chi(0)=0$, $\chi(1)=7/8$.

Resultat: $\{0,\ 7/12,\ 7/8\}^{max}_{min}=\begin{cases}7/8\\ 0\end{cases} \Rightarrow \forall x\in D:\ f(x)<1.$

Övning 4.6 (s.78)

$f(x,y)=4x^2y^2-2xy^4-3x^2$, $D=[0,2]^2=\{x:\ 0\le x,y\le 2\}$.



$\partial D=\gamma_1+\gamma_2+\gamma_3+\gamma_4$

(i) $\mathring{D}=]0,2[^2=\{(x,y):\ 0<x,y<2\}$.

$\forall x\in\mathring{D}:\ \frac{\partial f}{\partial x}=8xy^2-2y^4-6x,\ \frac{\partial f}{\partial y}=8x^2y-8xy^3;$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases}8xy(x-y^2)=0\\ 4xy^2-y^4-3x=0\end{cases} \Leftrightarrow \begin{cases}x=y^2\\ y^4-y^2=0\end{cases} \Leftrightarrow \begin{cases}x=1\\ y=1\end{cases};$$

$f(1,1)=-1$.

(ii) $\gamma_1=[0,2]\times\{0\}=\{(x,0):\ 0\le x\le 2\}$.

$f(x,0)\equiv 0$.

$\gamma_2=\{2\}\times[0,2]=\{(2,y):\ 0\le y\le 2\}$.

$f(2,y)=16y^2-4y^4-12=\phi(y)$, $0\le y\le 2$.

$\phi'(y)=32y-16y^3=-16y(y^2-2)=0 \Leftrightarrow y=\sqrt{2}$.

$\phi(0)=-12$, $\phi(\sqrt{2})=4$, $\phi(2)=-12$.

$\gamma_3 = [0,2]\times\{2\} = \{(x,2):\ 0\le x\le 2\}$.

$f(x,2) = 13x^2-32x = \psi(x)\ ,\ 0\le x\le 2$.

$\psi'(x) = 26x-32 = 0 \Leftrightarrow x = \frac{16}{13}$;

$\psi(0)=0,\ \psi(\frac{16}{13}) = -\frac{256}{13} \approx -19{,}69\ ,\ \psi(2) = -12$.

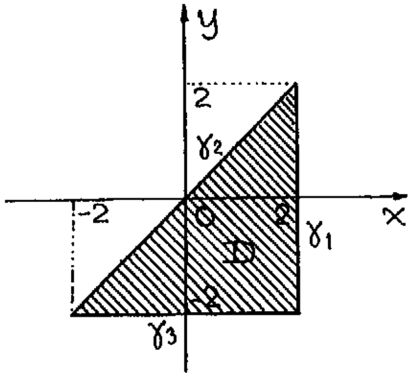
$\gamma_4 = \{0\}\times[0,2] = \{(0,y):\ 0\le y\le 2\}$.

$f(0,y) \equiv 0$.

Resultat: $\{0,-12,-\frac{256}{13},4\}^{max}_{min} = \begin{cases} 4 \\ -256/13 \end{cases}$

Övning 4.7 (s.79)

$f(x,y) = y^2+(x^2-1)y,\quad D = [0,2]^2 \setminus \{(x,y):\ y<x\}$.



(i) $\mathring{D} = \{(x,y):\ y<x,\ x>2,\ y>-2\}$.

$\forall x\in\mathring{D}:\ \frac{\partial f}{\partial x} = 2xy \wedge \frac{\partial f}{\partial y} = 2y+x^2-1$.

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow 2xy = 0 \wedge 2y+x^2-1 = 0 \Leftrightarrow (x,y) = (1,0)$;

$f(1,0) = 0$

(ii) $\gamma_1 = \{2\}\times[-2,2] = \{(2,y):\ -2\le y\le 2\}$.

$f(2,y) = y^2+3y = \phi(y),\ -2\le y\le 2$.

$\phi'(y) = 2y+3 = 0 \Leftrightarrow y = -\frac{3}{2}$.

$\phi(-2) = -2\ ,\ \phi(-\frac{3}{2}) = -\frac{9}{4},\ \phi(2) = 10$

$\gamma_2 = \{(x,y):\ y=x,\ -2\le x\le 2\}$.

$f(x,x) = x^3+x^2-x = \psi(x)\ ,\ -2\le x\le 2$.

$\psi'(x) = 3x^2+2x-1 = 0 \Leftrightarrow (x+1)(3x-1) = 0 \Leftrightarrow x = -1 \vee x = \frac{1}{3}$;

$\psi(-2) = -2,\ \psi(-1) = 1,\ \psi(\frac{1}{3}) = -\frac{5}{27},\ \psi(2) = 10$.

$\gamma_3 = [-2,2]\times\{-2\} = \{(x,-2):\ -2\le x\le 2\}$.

$f(x,-2) = 6-2x^2 = \chi(x),\ -2\le x\le 2$.

$\chi'(x) = -4x = 0 \Leftrightarrow x = 0$.

$\chi(-2) = -2,\ \chi(0) = 6,\ \chi(2) = -2$.

Resultat: $\{0,-2,10,1,-\frac{5}{27},-\frac{9}{4},6\}^{max}_{min} = \begin{cases} 10 \\ -9/4 \end{cases}$

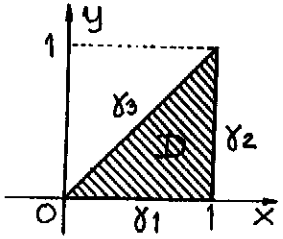
Anm. Läs nu författarnas lösning.

Övning 4.8 (s.79)

$0\le y\le x\le 1 \Leftrightarrow 0\le y\le x \wedge 0\le x\le 1$.

forts.

$f(x,y)=3x^2+y^3-3xy^2$, $D=\{(x,y)\in[0,1]^2 : y\le x\}$.



(i) $\mathring{D}=\{(x,y)\in\,]0,1[^2 : y<x\}$.

$\forall \mathbf{x}\in\mathring{D}$: $\frac{\partial f}{\partial x}=6x-3y^2 \wedge \frac{\partial f}{\partial y}=3y^2-6xy$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 2x-y^2=0 \\ y^2-2xy=0 \end{cases} \Leftrightarrow \begin{cases} 2x=y^2 \\ y^2-y^3=0 \end{cases};$$

Systemet saknar lösningar, så några stationära punkter i $\mathring{D}$ finns inte.

(ii) $\gamma_1=[0,1]\times\{0\}=\{(x,0): 0\le x\le 1\}$.

$f(x,0)=3x^2$ växande i $0\le x\le 1$.

$f(0,0)=0$, $f(1,0)=3$.

$\gamma_2=\{1\}\times[0,1]=\{(1,y): 0\le y\le 1\}$.

$f(1,y)=y^3-3y^2+3=\phi(y)$, $0\le y\le 1$

$\phi'(y)=3y^2-6y=3y(y-2)\le 0 \Rightarrow \phi$ avtagande.

$\phi(0)=3$, $\phi(1)=1$.

forts.

$\gamma_3=\{(x,y): y=x,\ 0\le x\le 1\}$.

$f(x,x)=3x^2-2x^3=\psi(x)$, $0\le x\le 1$.

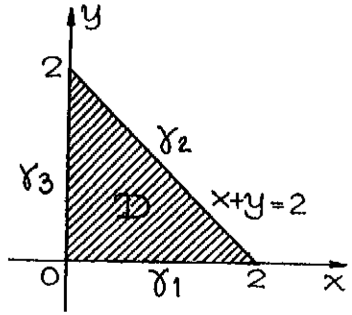
$\psi'(x)=6x-6x^2=6x(1-x)\ge 0$ ψ växande.

$\psi(0)=0$, $\psi(1)=1$.

Resultat: $\{0,3,1\}_{\min}^{\max}=\begin{cases}3\\0\end{cases}$.

Övning 4.9 (s. 79)

$f(x,y)=x+y-(x^2+y^2)^2$, $D=\{(x,y)\in[0,2]^2 : y<-x+2\}$.



(i) $\mathring{D}=\{(x,y)\in\,]0,2[^2 : y<-x+2\}$.

$\forall \mathbf{x}\in\mathring{D}$: $\frac{\partial f}{\partial x}=1-4x(x^2+y^2)$, $\frac{\partial f}{\partial y}=1-4y(x^2+y^2)$;

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 1-4x(x^2+y^2)=0 \\ 1-4y(x^2+y^2)=0 \end{cases} \Leftrightarrow \begin{cases} y=x \\ 1-8x^3=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \end{cases}.$$

$f(\frac{1}{2},\frac{1}{2})=\frac{3}{4}$.

(ii) $\gamma_1=[0,2]\times\{0\}=\{(x,0): 0\le x\le 2\}$.

forts.

$$f(x,0) = x - x^4 = \phi(x)\,,\quad 0 \le x \le 2.$$

$$\phi'(x) = 1 - 4x^3 = 0 \Leftrightarrow x = 4^{-1/3};$$

$$\underline{\phi(0) = 0},\ \underline{\phi(1/4^{1/3}) = 3/2^{8/3} \approx 0{,}47},\ \underline{\phi(2) = -14};$$

$$\underline{\gamma_2 = \{(x,y):\ y = -x+2,\ 0 \le x \le 2\}}.$$

$$f(x, 2-x) = 2 - (4 - 4x + 2x^2)^2 = \psi(x)\,,\quad 0 \le x \le 2.$$

$$\psi'(x) = 0 \Rightarrow -16(x-1)\underbrace{(x^2 - 2x + 2)}_{\neq 0} = 0 \Leftrightarrow x = 1\ ;$$

$$\underline{\psi(0) = -14},\ \underline{\psi(1) = -2},\ \underline{\psi(2) = -14}.$$

$$\underline{\gamma_3 = \{0\} \times [0,2] = \{(0,y):\ 0 \le y \le 2\}}.$$

$$f(0,y) = y - y^4 = \chi(y)\,,\quad 0 \le y \le 2.$$

$$\chi'(y) = 1 - 4y^3 = 0 \Rightarrow y = 2^{-2/3}.$$

$$\underline{\chi(0) = 0},\ \underline{\chi(2^{-2/3}) = 3/2^{8/3}},\ \underline{\chi(2) = -14}.$$

__Resultat__: $\left\{\frac{3}{4}, 0, 3/2^{8/3}, -14, -2\right\}^{max}_{min} = \begin{cases} 3/4 \\ -14 \end{cases}$

Övning 4.10 (s.79)

$$\underline{f(x,y) = 2 - 3x^2 + y^2 - 3\ln(1 + x^2 + y^2)\,,\ D = \{\mathbf{x} : |\mathbf{x}| \le 1\}}.$$

Området D är den vanliga enhetsdisken.

(i) $\forall \mathbf{x} \in \mathring{D} = \{\mathbf{x} : |\mathbf{x}| < 1\}:\ \frac{\partial f}{\partial x} = -6x - \frac{6x}{1+|\mathbf{x}|^2},\ \frac{\partial f}{\partial y} = 2y - \frac{6y}{1+|\mathbf{x}|^2}.$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow x + \frac{x}{1+|\mathbf{x}|^2} = 0 \wedge y - \frac{3y}{1+|\mathbf{x}|^2} = 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x(2+|\mathbf{x}|^2) = 0 \\ y(|\mathbf{x}|^2 - 2) = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \vee |\mathbf{x}|^2 = 2 \end{cases} \Leftrightarrow (x,y) = (0,0).$$

$$\underline{f(0,0) = 2}.$$

(ii) $\underline{\partial D = \{(x,y):\ x^2 + y^2 = 1\}}.$

$$f(x, \pm\sqrt{1-x^2}) = 3 - 4x^2 - 3\ln 2 = \phi(x)\,,\quad -1 \le x \le 1.$$

$$\phi'(x) = -8x = 0 \Leftrightarrow x = 0.$$

$$\underline{\phi(-1) = -1 - 3\ln 2},\ \underline{\phi(0) = 3 - 3\ln 2},\ \underline{\phi(1) = -1 - 3\ln 2}.$$

__Resultat__: $\{-1-3\ln 2, 3-3\ln 2, 2\}^{max}_{min} = \begin{cases} 2 \\ -1-3\ln 2 \end{cases}$

Övning 4.11 (s. 79)

$$\underline{f(x,y) = x^2 + x(y^2 - 1)\,,\quad D = \{(x,y) : x^2 + y^2 \le 1\}}.$$

(i) $\forall \mathbf{x} \in \mathring{D} = \{\mathbf{x} : |\mathbf{x}| < 1\}:\ \frac{\partial f}{\partial x} = 2x + y^2 - 1 \wedge \frac{\partial f}{\partial y} = 2xy;$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} xy = 0 \\ 2x + y^2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - 1 = 0 \\ y = 0 \end{cases} \Leftrightarrow (x,y) = (\tfrac{1}{2}, 0).$$

$$\underline{f(\tfrac{1}{2}, 0) = -\tfrac{1}{4}}.$$

(ii) $\underline{\partial D = \{(x,y):\ x^2 + y^2 = 1\}}.$

$$x^2 + y^2 = 1 \Leftrightarrow y = \pm\sqrt{1-x^2} \Rightarrow f(x, \pm\sqrt{1-x^2}) = x^2 - x^3 = \phi(x).$$

$$\phi'(x) = 2x - 3x^2 = 0 \Leftrightarrow x = 0 \vee x = \tfrac{2}{3}.$$

forts.

$\underline{\phi(-1)=2}$, $\underline{\phi(\tfrac{2}{3})=\tfrac{4}{27}}$, $\underline{\phi(1)=0}$;

Resultat: $\{-\tfrac{1}{4}, 2, \tfrac{4}{27}, 0\}^{max}_{min}=\begin{cases}2\\-1/4\end{cases}$.

Övning 4.12 (s.79)

$f(x,y)=(x^2+3y^2)e^{-(x^2+y^2)}$, $D=\{(x,y):\ x^2+y^2\le 4\}$.

(i) $\mathring{D}=\{(x,y):\ \sqrt{x^2+y^2}<2\}$.

$$\forall x\in\mathring{D}:\ \begin{cases}\dfrac{\partial f}{\partial x}=(2x-2x^3-6xy^2)e^{-(x^2+y^2)}\\ \dfrac{\partial f}{\partial y}=(6y-2x^2y-6y^3)e^{-(x^2+y^2)}\end{cases};$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0\Rightarrow\begin{cases}2x(1-x^2-3y^2)=0\\2y(3-x^2-3y^2)=0\end{cases}\Leftrightarrow\begin{cases}x=0\vee x^2+3y^2=1\\y=0\vee x^2+3y^2=3\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}x=0\\x^2+3y^2=3\end{cases}\vee\begin{cases}x^2+3y^2=1\\y=0\end{cases}\vee\begin{cases}x=0\\y=0\end{cases}\vee\begin{cases}x^2+3y^2=1\\x^2+3y^2=3\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}x=0\\y^2=1\end{cases}\vee\begin{cases}x^2=1\\y=0\end{cases}\vee\begin{cases}x=0\\y=0\end{cases}\Leftrightarrow\begin{cases}x=0\\y=\pm1\end{cases}\vee\begin{cases}x=\pm1\\y=0\end{cases}\vee\begin{cases}x=0\\y=0\end{cases}$$

$\underline{f(0,0)=0}$, $\underline{f(\pm1,0)=e^{-1}}$, $\underline{f(0,\pm1)=3e^{-1}}$.

(ii) $\partial D=\{(x,y):\ \sqrt{x^2+y^2}=1\}$.

$f(x,\pm\sqrt{2^2-x^2})=e^{-2}(12-2x^2)=\phi(x)$, $-2\le x\le 2$.

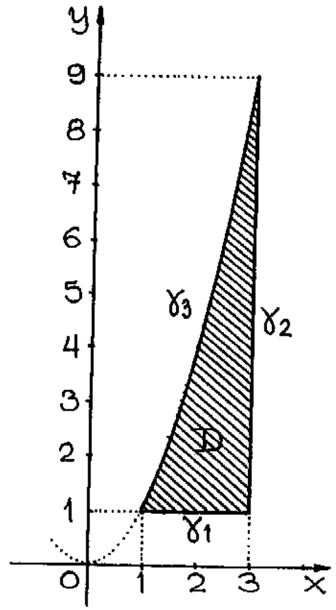
$\phi'(x)=-4e^{-2}x=0\Leftrightarrow x=0$.

forts.

$\underline{\phi(-2)=4e^{-2}}$, $\underline{\phi(0)=12e^{-2}}$, $\underline{\phi(2)=4e^{-2}}$

Resultat: $\{0, e^{-1}, 3e^{-1}, 4e^{-2}, 12e^{-2}\}^{max}_{min}=\begin{cases}3e^{-1}\\0\end{cases}$.

Övning 4.13 (s.79)



$f(x,y)=\dfrac{x^2+2y-4}{x^2y}$, $D=\{(x,y):\ 1\le y\le x^2,\ x\le 3\}$.

(i) $\mathring{D}=\{(x,y):\ 1<y<x^2,\ x<3\}$.

$$\forall x\in\mathring{D}:\ \begin{cases}\dfrac{\partial f}{\partial x}=\dfrac{2x\cdot x^2y-2xy(x^2+2y-4)}{x^4y^2}=-\dfrac{4(xy^2+2)}{x^4y^2}\\ \dfrac{\partial f}{\partial y}=\dfrac{2x^2y-x^2(x^2+2y-4)}{x^4y^2}=\dfrac{(4-x^2)x^2}{x^4y^2}=\dfrac{4-x^2}{x^2y^2}\end{cases}.$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0\Rightarrow\begin{cases}xy^2+2=0\\x^2-4=0\end{cases}\Leftrightarrow\begin{cases}x=2\\y^2+1=0\end{cases};\ \text{kritiska saknas.}$$

(ii) $\gamma_1 = [1,2]\times\{1\} = \{(x,1) : 1 \le x \le 3\}$.

$f(x,1) = 1 - \frac{2}{x^2} = \phi(x)\ ,\ 1 \le x \le 3\ ;$

$\phi'(x) = \frac{4}{x^3} > 0 \Rightarrow \phi$ strängt växande.

$\phi(1) = -1,\ \phi(3) = \frac{7}{9}$.

$\gamma_2 = \{3\}\times[1,9] = \{(3,y) : 1 \le y \le 9\}$.

$f(3,y) = \frac{5+2y}{9y} = \psi(y)\ ,\ 1 \le y \le 9\ ;$

$\psi'(y) = -\frac{5}{9y^2} < 0 \Rightarrow \psi$ strängt avtagande.

$\psi(1) = \frac{7}{9}\ ,\ \psi(9) = \frac{23}{81}$.

$\gamma_3 = \{(x,x^2) : 1 \le x \le 3\}$.

$f(x,x^2) = \frac{3x^2-4}{x^4} = \chi(x)\ ,\ 0 \le x \le 3$.

$\chi'(x) = \frac{6x\cdot x^4 - 4x^3\cdot(3x^2-4)}{x^8} = \frac{16-6x^2}{x^5} = 0 \Leftrightarrow x = \frac{4}{\sqrt{6}}$

$\chi(1) = -1,\ \chi\left(\frac{4}{\sqrt{6}}\right) = \frac{9}{16}\ ,\ \chi(3) = \frac{7}{9}$.

Resultat: $\left\{-1, \frac{7}{9}, \frac{23}{81}, \frac{9}{16}\right\}^{\max}_{\min} = \begin{cases} 7/9 \\ -1 \end{cases}$

Övning 4.14 (s. 79)

$f(x,y,z) = x^2+2yz\ ,\ K = \{(x,y,z) : x^2+y^2+z^2 \le 1\}$.

(i) $\mathring{K} = \{(x,y,z) : x^2+y^2+z^2 < 1\}$.

$\forall \mathbf{x} \in \mathring{K} :\ \frac{\partial f}{\partial x} = 2x \wedge \frac{\partial f}{\partial y} = z \wedge \frac{\partial f}{\partial z} = y\ ;$

forts.

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Leftrightarrow x = y = z = 0\ ;\quad f(0,0,0) = 0.$

(ii) $\partial K = \{(x,y,z) : x^2+y^2+z^2 = 1\}$.

$x^2+y^2+z^2 = 1 \Leftrightarrow x = \pm\sqrt{1-y^2-z^2}\ ;$

$f(\pm\sqrt{1-y^2-z^2}, y, z) = 1-y^2-z^2+2yz = g(z,y),\ D : y^2+z^2 \le 1.$

$g(y,z) = 1-y^2-z^2+2yz \Rightarrow \begin{cases} \frac{\partial g}{\partial y} = -2y+2z \\ \frac{\partial g}{\partial z} = -2z+2y \end{cases} ;\ (y,z) \in \mathring{D}.$

$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial z} = 0 \Leftrightarrow y = z\ ;\quad g(y,y) = 1.$

$y^2+z^2 = 1 \Leftrightarrow z = \pm\sqrt{1-y^2} \Rightarrow g(y, \pm\sqrt{1-y^2}) = \pm 2y\sqrt{1-y^2} = h(y).$

$h(\cos t, \sin t) = \pm \sin 2t \in [-1,1]\ ;$ (på randen)

Resultat: $\max_{\mathbf{x}\in K}\{f(\mathbf{x})\} = 1,\ \min_{\mathbf{x}\in K}\{f(\mathbf{x})\} = -1.$

Övning 4.15 (s. 80)

$f(x,y,z) = xyz+xy\ ,\ K = \{\mathbf{x} : |\mathbf{x}| \le 1,\ x,y,z \ge 0\}$.

(i) $\mathring{K} = \{\mathbf{x} : |\mathbf{x}| < 1,\ x,y,z > 0\}$.

$\forall \mathbf{x} \in \mathring{K} :\ \frac{\partial f}{\partial x} = yz+y \wedge \frac{\partial f}{\partial y} = xz+x \wedge \frac{\partial f}{\partial z} = xy \quad (*)$

$x,y,z > 0 \overset{(*)}{\Rightarrow}$ kritiska punkter saknas.

(ii) $x = 0 \Rightarrow f(\mathbf{x}) \equiv 0.$

$y = 0 \Rightarrow f(\mathbf{x}) \equiv 0.$

forts.

$z=0 \Rightarrow f(\mathbf{x}) = xy = g(x,y)\,,\quad x^2+y^2\leq 1,\ x\geq 0, y\geq 0$

$g(r\cos\theta, r\sin\theta) = \frac{1}{2}r^2\sin 2\theta = \tilde{g}(r,\theta)\,,\quad (r,\theta)\in[0,1]\times[0,\frac{\pi}{2}].$

$\max\{\tilde{g}(r,\theta)\} = \frac{1}{2}\,,\quad \min\{\tilde{g}(r,\theta)\} = 0.$

På den krökta delen av randen inför vi sfäriska koordinater, (Ex. 17, s. 26 i boken).

$f(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \sin^2\theta\cos\theta\sin\phi\cos\phi +$

$+\sin^2\theta\sin\phi\cos\phi = \sin^2\theta(1+\cos\theta)\sin\phi\cos\phi \Leftrightarrow$

$\Leftrightarrow F(\theta,\phi) = \sin^2\theta(1+\cos\theta)\underbrace{\sin\phi\cos\phi}_{\frac{1}{2}\sin 2\phi\in[0,\frac{1}{2}]}\,,\quad (\theta,\phi)\in[0,\frac{\pi}{2}]^2.$

$\psi(\theta) = \sin^2\theta(1+\cos\theta)\,;$ (Bestäm $\max\limits_{0\leq\theta\leq\frac{\pi}{2}}\psi(\theta)$).

$\psi'(\theta) = \sin 2\theta(1+\cos\theta) - \sin^3\theta =$

$= \sin\theta(2\cos\theta + 2\cos^2\theta - \sin^2\theta) =$

$= \sin\theta(3\cos^2\theta + 2\cos\theta - 1) =$

$= \sin\theta(3\cos\theta - 1)(\cos\theta + 1) = 0 \overset{(*)}{\Leftrightarrow} \cos\theta = \frac{1}{3}$

$F(\theta,\phi) = \frac{1}{2}\sin 2\phi\,(1-\cos^2\theta)(1+\cos\theta) \leq \frac{1}{2}\cdot 1\cdot\frac{8}{9}\cdot\frac{4}{3} = \frac{16}{27}.$

Resultat: $\max\limits_{\mathbf{x}\in K}\{f(\mathbf{x})\} = \frac{16}{27}\,,\quad \min\limits_{\mathbf{x}\in K}\{f(\mathbf{x})\} = 0.$

(*) Vi betraktar $\theta\in[0,\frac{\pi}{2}]$ s.a. $\cos\theta + 1 > 0$.

Anm. max antas i punkten $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$.

Optimering på icke-kompakta områden

Övning 4.16 (s. 80)

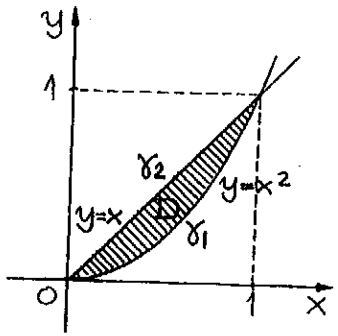
$f(x,y) = (y-x)e^{x^2-y}.$

a) $\frac{\partial f}{\partial x} = (2xy - 2x^2 - 1)e^{x^2-y}\,;\quad \frac{\partial f}{\partial y} = (x-y+1)e^{x^2-y}.$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 2xy-2x^2-1=0 \\ x-y+1=0 \end{cases} \Leftrightarrow \begin{cases} x=y-1 \\ 2y(y-1)-2(y-1)^2-1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x=y-1 \\ 2\cancel{y^2}-2y-\cancel{2y^2}+4y-2-1=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{3}{2} \end{cases} \Rightarrow (x,y)=(\frac{1}{2},\frac{3}{2}).$

b)



$D = \{(x,y): x^2\leq y\leq x\}\,;\quad \mathring{D} = \{(x,y): x^2<y<x\}.$

(i) $(\frac{1}{2},\frac{3}{2})\notin\mathring{D}.$

(ii) $\gamma_1 = \{(x,x^2): 0\leq x\leq 1\}.$

$f(x,x^2) = x^2 - x = \phi(x)\,,\quad 0\leq x\leq 1.$

$\phi'(x) = 2x-1 = 0 \Leftrightarrow x=\frac{1}{2}\,;\quad \phi(0)=\phi(1)=0,\ \phi(\frac{1}{2}) = -\frac{1}{4}.$

$\gamma_2 = \{(x,x): 0 \le x \le 1\}$.

$f(x,x) \equiv 0$.

Resultat: $\max\limits_{x \in D}\{f(x)\} = 0$, $\min\limits_{x \in D}\{f(x)\} = -\frac{1}{4}$.

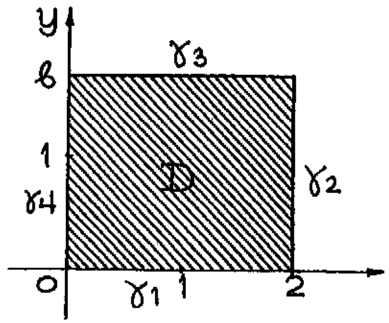
c) $\lim\limits_{x\to\infty} f(x,x^2) = \lim\limits_{x\to\infty}(x^2 - x) = \infty$.

Max saknas i området $y \ge x^2$.

Läs nu författarnas lösning.

Övning 4.17 (s.80)

a) $f(x,y) = xy^2e^{-xy}$, $D = \{(x,y): 0 \le x \le 2,\ 0 \le y \le b\}$.



(i) $\mathring{D} =]0,2[\times]0,b[= \{(x,y): 0<x<2,\ 0<y<b\}$.

$\forall x \in \mathring{D}: \frac{\partial f}{\partial x} = (y^2 - xy^3)e^{-xy} \wedge \frac{\partial f}{\partial y} = (2xy - x^2y^2)e^{-xy}$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} y^2(1-xy)=0 \\ xy(2-xy)=0 \end{cases} \Leftrightarrow xy = 2 \vee xy = 1.$$

$xy = 2 \Rightarrow f(x,\frac{2}{x}) = \frac{4}{x}e^{-2}$; $xy = 1 \Rightarrow f(x,\frac{1}{x}) = \frac{1}{x}e^{-1}$;

f har inga extrema i $\mathring{D}$.

(ii) $\gamma_1 = [0,2]\times\{0\} = \{(x,0): 0 \le x \le 2\}$.

$f(x,0) \equiv 0$.

$\gamma_2 = \{2\}\times[0,b] = \{(2,y): 0 \le y \le b\}$.

$f(2,y) = 2y^2e^{-2y} = \phi(y)$, $0 \le y \le b$.

$\phi'(y) = 4y(1-y)e^{-2y} = 0 \Leftrightarrow y = 1$;

$\phi(0) = 0$, $\phi(1) = 2e^{-2}$, $\phi(b) = 2b^2e^{-2b}$.

$\gamma_3 = [0,2]\times\{b\} = \{(x,b): 0 \le x \le 2\}$.

$f(x,b) = b^2xe^{-bx} = \psi(x)$, $0 \le x \le 2$.

$\psi'(x) = b^3(\frac{1}{b} - x)e^{-bx} = 0 \Leftrightarrow x = \frac{1}{b}$;

$\psi(0) = 0$, $\psi(\frac{1}{b}) = be^{-1}$, $\psi(2) = 2b^2e^{-2b}$.

$\gamma_4 = \{0\}\times[0,b] = \{(0,y): 0 \le y \le b\}$.

$f(0,y) \equiv 0$.

Resultat: $\min\limits_{x\in D}\{f(x)\} = 0$, $\max\limits_{x\in D}\{f(x)\} = be^{-1}$.

f saknar max i $0 \le x \le 2, y \ge 0$, ty $\lim\limits_{b\to\infty} be^{-1} = \infty$.

Övning 4.18 (s.80)

$f(x,y) = x^2ye^{-(x^2+2y^2)}$, $(x,y) \in \mathbb{R}^2$.

forts.

$\frac{\partial f}{\partial x} = 2xy(1-x^2)e^{-(x^2+2y^2)}$; $\frac{\partial f}{\partial y} = x^2(1-4y^2)e^{-(x^2+2y^2)}$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 2xy(1-x^2)=0 \\ x^2(1-4y^2)=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \vee y=0 \vee x=\pm 1 \\ x=0 \vee y=\pm\frac{1}{2} \end{cases} \Leftrightarrow$$

$\Leftrightarrow \mathbf{x}=(0,0) \vee \mathbf{x}=(0,\frac{1}{2}) \vee \mathbf{x}=(0,-\frac{1}{2}) \vee \mathbf{x}=(1,\frac{1}{2}) \vee \mathbf{x}=(-1,\frac{1}{2})$

$\vee\ \mathbf{x}=(1,-\frac{1}{2}) \vee \mathbf{x}=(-1,-\frac{1}{2})$.

$\underline{f(0,0)=f(0,\frac{1}{2})=f(0,-\frac{1}{2})=0}$, $\underline{f(1,\frac{1}{2})=f(-1,\frac{1}{2})=\frac{1}{2}e^{-3/2}}$;

$\underline{f(1,-\frac{1}{2})=f(-1,-\frac{1}{2})=-\frac{1}{2}e^{-3/2}}$.

$\sqrt{x^2+y^2}>2 \Rightarrow |f(\mathbf{x})| = x^2|y|e^{-(x^2+2y^2)} \leq |\mathbf{x}|^3 e^{-|\mathbf{x}|^2}$

$\lim\limits_{|\mathbf{x}|\to\infty} |f(\mathbf{x})| \leq \lim\limits_{|\mathbf{x}|\to\infty} |\mathbf{x}|^3 e^{-|\mathbf{x}|^2} = 0 \Leftrightarrow \lim\limits_{|\mathbf{x}|\to\infty} f(\mathbf{x}) = 0.$

<u>Resultat</u>: $\max\limits_{\mathbf{x}\in\mathbb{R}^2}\{f(\mathbf{x})\} = \frac{1}{2}e^{-3/2}$, $\min\limits_{\mathbf{x}\in\mathbb{R}^2}\{f(\mathbf{x})\} = -\frac{1}{2}e^{-3/2}$.

Övning 4.19 (s. 80)

$\underline{f(x,y)=(x^2+y)e^{-(x+y)},\ D=[0,\infty[^2}$.

(i) $\forall \mathbf{x}\in\,]0,\infty[^2$: $\frac{\partial f}{\partial x} = (2x-x^2-y)e^{-(x+y)}$, $\frac{\partial f}{\partial y} = (1-x^2-y)e^{-(x+y)}$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 2x-x^2-y=0 \\ 1-x^2-y=0 \end{cases} \Leftrightarrow \begin{cases} 2x=1 \\ y=1-x^2 \end{cases} \Leftrightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{3}{4} \end{cases};$$

$\underline{f(\frac{1}{2},\frac{3}{4})=e^{-5/4}}$.

(ii) $f(0,y)=ye^{-y}=\phi(y) \Rightarrow \phi'(y)=(1-y)e^{-y}=0 \Leftrightarrow y=1.$

$\underline{\phi(0)=0}$, $\underline{\phi(1)=e^{-1}}$, $\underline{\phi(\infty)=0}$.

$f(x,0)=x^2e^{-x}=\psi(x) \Rightarrow \psi'(x)=x(2-x)e^{-x}=0 \Leftrightarrow x=2$;

$\underline{\psi(0)=0}$, $\underline{\psi(2)=4e^{-2}}$, $\underline{\psi(\infty)=0}$.

(iii) $\forall \mathbf{x}\in D$: $f(\mathbf{x})\geq 0$.

$\lim\limits_{|\mathbf{x}|\to\infty} f(\mathbf{x}) = \lim\limits_{|\mathbf{x}|\to\infty} (x^2+y)e^{-(x+y)}\ [y=kx] =$

$= \lim\limits_{\to\infty} (x^2+kx)e^{-(k+1)x} = 0.$

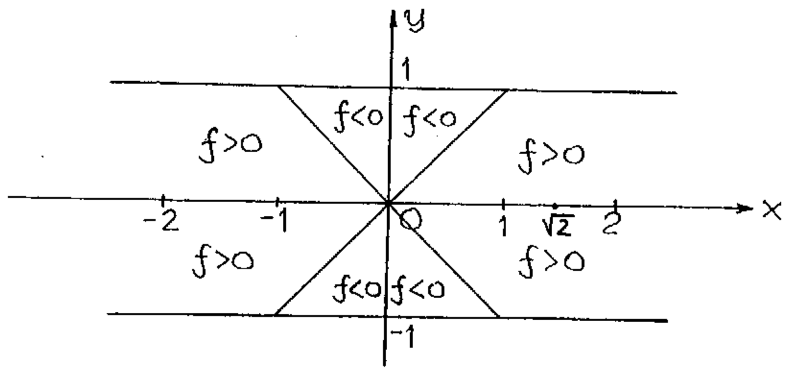
<u>Resultat</u>: $\max\limits_{\mathbf{x}\in D}\{f(\mathbf{x})\} = 4e^{-2}$, $\min\limits_{\mathbf{x}\in D}\{f(\mathbf{x})\} = 0$.

Övning 4.20 (s. 80)

$\underline{f(x,y) = \frac{x^2-y^2}{(\ +x^2+y^2)^2},\ -1\leq y\leq 1.}$

$f(-x,y)=f(x,y)$: symmetri m.a.p. yz-planet.

$f(x,-y)=f(x,y)$: — " — — " — xz-planet.



Vi studerar f:s restriktion till $D=[0,\omega]\times[-1,1]$.

(i) $\underline{\mathring{D} =]0,\omega[\times]-1,1[= \{(x,y):\ 0<x<\omega,\ -1<y<1\}}$. (*)

$\frac{\partial f}{\partial x} = \frac{2x(2-x^2+3y^2)}{(2+x^2+y^2)^2}$, $\frac{\partial f}{\partial y} = \frac{-2y(2+3x^2-y^2)}{(2+x^2+y^2)^2}$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} x(2-x^2+3y^2)=0 \\ y(2+3x^2-y^2)=0 \end{cases} \Leftrightarrow \begin{cases} x=0=2-x^2+3y^2 \\ y=0=2+3x^2-y^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \vee \begin{cases} x=0 \\ 2-y^2=0 \end{cases} \vee \begin{cases} 2-x^2=0 \\ y=0 \end{cases} \overset{(*)}{\Leftrightarrow} \begin{cases} x=\sqrt{2} \\ y=0. \end{cases} \quad (\omega>\sqrt{2}).$$

$\underline{f(\sqrt{2},0) = \frac{1}{8}}$.

(ii) $\underline{\gamma_1 = [0,\omega]\times\{1\} = \{(x,1):\ 0\le x\le\omega\}}$;

$f(x,0) = \frac{x^2-1}{(3+x^2)^2} = \phi(x)$, $0\le x\le\omega$.

$\phi'(x) = \frac{2x(3+x^2)^2-4x(3+x^2)(x^2-1)}{(3+x^2)^4} = \frac{2x(5-x^2)}{(3+x^2)^3} = 0 \Rightarrow x=\sqrt{5}$.

$\underline{\phi(0) = -\frac{1}{9}}$, $\underline{\phi(\sqrt{5}) = \frac{1}{16}}$, $\phi(\omega) = \frac{\omega^2-1}{(3+\omega^2)^2} \xrightarrow[\omega\to\infty]{} 0$.

$\underline{\gamma_2 = \{0\}\times[-1,1] = \{(0,y):\ -1\le y\le 1\}}$.

$f(0,y) = -\frac{y^2}{(2+y^2)^2} = \psi(y)$, $-1\le y\le 1$.

$\psi'(y) = -\frac{2y(2+y^2)^2-4y^3(2+y^2)}{(2+y^2)^4} = -\frac{2y(y^2-2)}{(2+y^2)^2} = 0 \Rightarrow y=0$

$\underline{\psi(-1) = -\frac{1}{9}}$, $\underline{\psi(0)=0}$, $\underline{\psi(1) = -\frac{1}{9}}$.

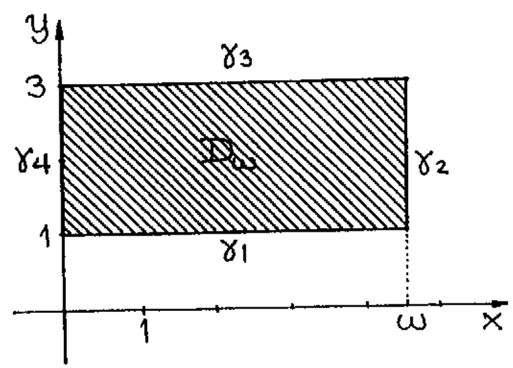
$\underline{\gamma_3 = [0,\omega]\times\{-1\}}$ ger samma resultat som γ_1.

<u>Resultat</u>: $\{\frac{1}{8}, -\frac{1}{9}, \frac{1}{16}, 0\}^{max}_{min} = \begin{cases} 1/8 \\ -1/9 \end{cases}$

Övning 4.21 (s. 80)

$\underline{f(x,y) = xe^{-xy}}$, $\underline{D = [0,\infty[\times[1,3]}$.

Vi betraktar restriktionen till $[0,\omega]\times[1,3]$.



(i) $\underline{\mathring{D}_\omega =]0,\omega[\times]1,3[= \{(x,y):\ 0<x<\omega,\ 1<y<3\}}$.

$\frac{\partial f}{\partial x} = (1+xy)e^{-x} > 0 \ \wedge\ \frac{\partial f}{\partial y} = -x^2e^{-xy} < 0$.

Stationära punkter saknas i D_ω.

(ii) $\underline{\gamma_1 = [0,\omega]\times\{1\} = \{(x,1):\ 0\le x\le\omega\}}$.

$f(x,1) = xe^{-x} = \phi(x)$, $0\le x\le\omega$;

$\phi'(x) = (1-x)e^{-x} = 0 \Leftrightarrow x=1$;

$\underline{\phi(0)=0}$, $\underline{\phi(1)=e^{-1}}$, $\phi(\omega) = \omega e^{-\omega} \xrightarrow[\omega\to\infty]{} 0$.

$\underline{\gamma_2 = \{\omega\}\times[1,3] = \{(\omega,y):\ 1\le y\le 3\}}$.

$f(\omega,y) = \omega e^{-\omega y} = \psi(y)$, $1\le y\le 3$ (avtagande).

$\psi(1) = \omega e^{-\omega} \xrightarrow[\omega\to\infty]{} 0$, $\psi(3) = \omega e^{-3\omega} \xrightarrow[\omega\to\infty]{} 0$.

$\gamma_3 = [0,\omega]\times\{3\} = \{(x,3):\ 0\le x\le\omega\}$.

$f(x,3) = xe^{-3x} = \chi(x),\quad 0\le x\le \omega$;

$\chi'(x) = (1-3x)e^{-3x} = 0 \Leftrightarrow x = \frac{1}{3}$.

$\chi(0) = 0,\ \chi(\frac{1}{3}) = \frac{1}{3}e^{-1},\ \chi(\omega) = \omega e^{-3\omega} \xrightarrow[\omega\to\infty]{} 0$.

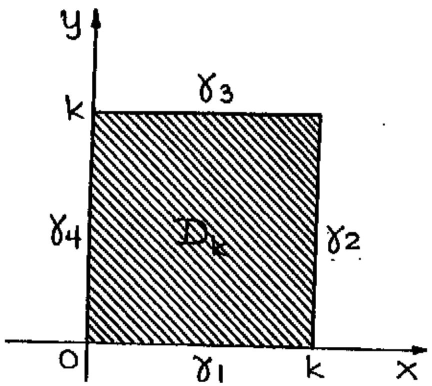
$\gamma_4 = \{0\}\times[1,3] = \{(0,y):\ 1\le y\le 3\}$.

$f(0,y)\equiv 0$.

Resultat: $\max\limits_{x\in D}\{f(x)\} = e^{-1}$.

Övning 4.22 (s. 81)

$f(x,y) = (x+y+1)^3 - 27xy,\quad D = [0,\infty[^2$.



(i) $\mathring{D}_k =]0,k[^2 = \{(x,y):\ 0<x,y<k\}$.

$\frac{\partial f}{\partial x} = 3(x+y+1)^2 - 27y,\ \frac{\partial f}{\partial y} = 3(x+y+1)^2 - 27x$;

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \Rightarrow y = x \wedge 3(2x+1)^2 - 27x = 0 \Rightarrow$ forts.

$\Leftrightarrow \begin{cases}(2x+1)^2 - 9x = 0\\ y = x\end{cases} \Leftrightarrow \begin{cases}4x^2 - 5x + 1 = 0\\ y = x\end{cases} \Leftrightarrow \begin{cases}(x-1)(4x-1) = 0\\ y = x\end{cases}$

$\Leftrightarrow (x,y) = (\frac{1}{4},\frac{1}{4}) \vee (x,y) = (1,1)$.

$f(\frac{1}{4},\frac{1}{4}) = \frac{27}{16},\quad f(1,1) = 0$.

(ii) $\gamma_1 = [0,k]\times\{0\} = \{(x,0):\ 0\le x\le k\}$.

$f(x,0) = (x+1)^3 = \phi(x),\ 0\le x\le k$, strängt växande.

$\phi(0) = 1,\ \phi(k) = (k+1)^3 \xrightarrow[k\to\infty]{} \infty$.

$\gamma_2 = \{k\}\times[0,k] = \{(k,y):\ 0\le y\le k\}$.

$f(k,y) = (1+k+y)^3 - 27ky = \psi(y),\ 0\le y\le k$.

$\psi'(y) = 3(1+k+y)^2 - 27k = 0 \Leftrightarrow y = 3\sqrt{k} - k - 1 = \sqrt{k} - (\sqrt{k}-1)^2$

För $k>3$ blir $3\sqrt{k} - k - 1 < 0$. Vi är intresserade av stora k.

$\gamma_3 = [0,k]\times\{k\} = \{(x,k):\ 0\le x\le k\}$.

$f(x,k) = (1+k+x)^3 - 27kx$ samma som i γ_2.

$\gamma_4 = \{0\}\times[0,k] = \{(0,y):\ 0\le y\le k\}$.

$f(0,y) = (y+1)^3,\ 0\le y\le k$, samma som i γ_1.

$\forall x\in D_k:\ f(x)\ge 1 > 0 \Leftrightarrow (x+y+1)^3 - 27xy > 0 \Leftrightarrow$

$\Leftrightarrow 27xy < (x+y+1)^3 \Leftrightarrow xy < \frac{(x+1+1)^3}{3^3} \Leftrightarrow \sqrt[3]{xy} < \frac{x+y+1}{3}$.

Optimering med bivillkor

Övning 4.23 (s. 81)

$f(x,y)=(2x+3y+1)^2, \quad D=\{(x,y): x^2+y^2=1\}.$

$$\begin{cases} \operatorname{grad} f(x,y)=(4(2x+3y+1),\, 6(2x+3y+1)). \\ \operatorname{grad} g(x,y)=(2x,2y)\;;\; (g(x,y)=x^2+y^2-1=0). \end{cases}$$

$$\begin{vmatrix} 4(2x+3y+1) & 2x \\ 6(2x+3y+1) & 2y \end{vmatrix} = 2(2x+3y+1)\begin{vmatrix} 2 & 2x \\ 3 & 2y \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow 4(2x+3y+1)(2y-3x)=0 \Leftrightarrow 2y-3x=0 \Leftrightarrow y=\tfrac{3}{2}x\;;$$

$$\begin{cases} x^2+y^2=1 \\ y=\frac{3}{2}x \end{cases} \Leftrightarrow \begin{cases} \frac{13}{4}x^2=1 \\ y=\frac{3}{2}x \end{cases} \Leftrightarrow \begin{cases} x=\frac{2}{\sqrt{13}} \\ y=\frac{3}{\sqrt{13}} \end{cases} \vee \begin{cases} x=-\frac{2}{\sqrt{13}} \\ y=-\frac{3}{\sqrt{13}} \end{cases}\;;$$

$$f\left(\tfrac{2}{\sqrt{13}}, \tfrac{3}{\sqrt{13}}\right)=(\sqrt{13}+1)^2, \quad f\left(-\tfrac{2}{\sqrt{13}}, -\tfrac{3}{\sqrt{13}}\right)=(\sqrt{13}-1)^2\;;$$

Det är uppenbart att $f(x,y)\geq 0$ och att $f(x_0,y_0)=0$ för de (x_0,y_0) som löser systemmet $f(x,y)=0=g(x,y)$.

Resultat: $\{(2x+3y+1)^2 : x^2+y^2=1\}^{\max}_{\min} = \begin{cases} (\sqrt{13}+1)^2 \\ 0 \end{cases}.$

Övning 4.24 (s. 81)

$f(x,y)=\dfrac{x+y-10}{\sqrt{2}}, \quad g(x,y)=x^2-xy+2y^2-1=0\;;$

$\operatorname{grad} f(x)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), \quad \operatorname{grad} g(x)=(2x-y,\, 4y-x)\;;$

$$\begin{vmatrix} 1/\sqrt{2} & 2x-y \\ 1/\sqrt{2} & -x+4y \end{vmatrix} = \frac{1}{\sqrt{2}}(-x+4y+y-2x)=\frac{1}{\sqrt{2}}(5y-3x)=0 \Leftrightarrow$$

$$\Leftrightarrow 5y=3x \Leftrightarrow y=\tfrac{3}{5}x\;;$$

$$\begin{cases} y=\frac{3}{5}x \\ x^2-xy+2y^2=1 \end{cases} \Leftrightarrow \begin{cases} y=\frac{3}{5}x \\ x^2=\frac{25}{28} \end{cases} \Leftrightarrow \begin{cases} x=\frac{5}{2\sqrt{7}} \\ y=\frac{3}{2\sqrt{7}} \end{cases} \vee \begin{cases} x=-\frac{5}{2\sqrt{7}} \\ y=-\frac{3}{2\sqrt{7}} \end{cases}\;;$$

$$f\left(\tfrac{5}{2\sqrt{7}},\tfrac{3}{2\sqrt{7}}\right)=\frac{4-10\sqrt{7}}{\sqrt{14}}, \quad f\left(-\tfrac{5}{2\sqrt{7}},-\tfrac{3}{2\sqrt{7}}\right)=-\frac{4+10\sqrt{7}}{\sqrt{14}}$$

Resultat: Kortaste avståndet mellan linjen $x+y=10$ och ellipsen $x^2-xy+2y^2=1$ är

$$\left|f\left(\tfrac{5}{2\sqrt{7}},\tfrac{3}{2\sqrt{7}}\right)\right| = \frac{10\sqrt{7}-4}{\sqrt{14}} \approx 6{,}00 \text{ le}.$$

Övning 4.25 (s. 81)

$f(x,y)=x^2+y^2, \quad g(x,y)=13x^2+13y^2+10xy-72=0.$

$\frac{\partial f}{\partial x}=2x, \ \frac{\partial f}{\partial y}=2y\;;\ \frac{\partial g}{\partial x}=26x+10y, \ \frac{\partial g}{\partial y}=26y+10x\;;$

$$\begin{vmatrix} 2x & 26x+10y \\ 2y & 26y+10x \end{vmatrix} = \begin{vmatrix} 2x & 10y \\ 2y & 10x \end{vmatrix} = 20(x-y)(x+y)=0 \Leftrightarrow$$

$$\begin{cases} x=y \Rightarrow 36x^2=72 \Leftrightarrow x=\pm\sqrt{2} \\ x=-y \Rightarrow 16x^2=72 \Leftrightarrow x=\pm\frac{3\sqrt{2}}{2} \end{cases} \Leftrightarrow \begin{cases} x=\pm(\sqrt{2},\sqrt{2}) \\ x=\pm(\sqrt{2},-\sqrt{2}) \end{cases}\;;$$

$f(\sqrt{2},\sqrt{2})=f(-\sqrt{2},-\sqrt{2})=4$; $g(\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2})=g(-\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2})=9$.

Resultat: Det största avståndet är 3 och det minsta 2. Ellipsen finns uppritad i facit, s. 87.

Övning 4.26 (s. 81)

$f(x,y)=x^2y$, $g(x,y)=x+y-1=0$.

$\operatorname{grad} f(x,y)=(2xy,x^2)$; $\operatorname{grad} g(x,y)=(1,1)$. (*)

$$\begin{vmatrix} 2xy & 1 \\ x^2 & 1 \end{vmatrix} = 2xy-x^2=x(2y-x)=0 \Leftrightarrow x=0 \vee x=2y;$$

(i) $x=0 \overset{(*)}{\Rightarrow} y=1 \Rightarrow (x,y)=(0,1)$;

(ii) $x=2y \overset{(*)}{\Rightarrow} 3y=1 \Leftrightarrow y=\frac{1}{3} \Rightarrow (x,y)=(\frac{2}{3},\frac{1}{3})$;

Ingen av dessa två punkter ger extremum, ty

$$y=1-x \Rightarrow f(x,1-x)=x^2(1-x) \xrightarrow[x\to-\infty]{x\to+\infty} \begin{cases}-\infty\\+\infty\end{cases}.$$

Svar: Nej, det finns inte.

Övning 4.27 (s. 81)

$f(x,y,z)=xy\sqrt{z}$, $g(x,y,z)=x+y+z-1=0$, $x,y,z>0$.

$$\begin{cases} x+y+z=1 \Leftrightarrow z=1-x-y; \\ f(x,y,1-x-y)=xy\sqrt{1-x-y}=\phi(x,y); \quad x+y<1,\ x,y>0. \end{cases}$$

$$\frac{\partial\phi}{\partial x}=y\sqrt{1-x-y}-\frac{xy}{2\sqrt{1-x-y}}=\frac{2y(1-x-y)-xy}{2\sqrt{1-x-y}}=\frac{2y-3xy-2y^2}{2\sqrt{1-x-y}};$$

$$\frac{\partial\phi}{\partial y}=x\sqrt{1-x-y}-\frac{xy}{2\sqrt{1-x-y}}=\frac{2x(1-x-y)-xy}{2\sqrt{1-x-y}}=\frac{2x-3xy-2x^2}{2\sqrt{1-x-y}};$$

$\frac{\partial\phi}{\partial x}=0=\frac{\partial\phi}{\partial y} \Rightarrow y=x$ (p.g.a. symmetrin).

$\frac{\partial\phi}{\partial x}=0 \wedge y=x \Leftrightarrow 2y-3xy-2y^2=0 \wedge y=x \Leftrightarrow 2x-5x^2=0 \wedge y=x \Leftrightarrow x=y=\frac{2}{5} \Rightarrow (x,y)=(\frac{2}{5},\frac{2}{5})$. (*)

$x+y+z=1 \overset{(*)}{\Rightarrow} z=1-x-y=\frac{1}{5}$; $(x,y,z)=(\frac{2}{5},\frac{2}{5},\frac{1}{5})$.

$$f(\tfrac{2}{5},\tfrac{2}{5},\tfrac{1}{5})=\frac{4\sqrt{5}}{125}=\max_{x\in D}\{f(x)\}.$$

Anm. $D=\{(x,y,z)\in\mathbb{R}^3_+ : x+y+z=1\}$.

Övning 4.28 (s. 81)

$f(x,y,z)=2x\cdot 2y\cdot 2z=8xyz$; (målfunktionen)

$g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$; (bivillkor) (*)

$$\operatorname{grad} f(x) \parallel \operatorname{grad} g(x) \Rightarrow \frac{8yz}{2x/a^2}=\frac{8xz}{2y/b^2}=\frac{8xy}{2z/c^2} \Leftrightarrow$$

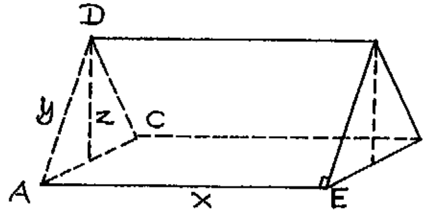
$$\Leftrightarrow \frac{8xyz}{2x^2/a^2}=\frac{8xyz}{2y^2/b^2}=\frac{8xyz}{2z^2/c^2} \Leftrightarrow \frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=1 \overset{(*)}{\Leftrightarrow}$$

$$\Leftrightarrow 3\frac{x^2}{a^2}=3\frac{y^2}{b^2}=3\frac{z^2}{c^2}=1 \Leftrightarrow x=\frac{a}{\sqrt{3}} \wedge y=\frac{b}{\sqrt{3}} \wedge z=\frac{c}{\sqrt{3}};$$

$$\left\{8xyz : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\right\}_{max}=8\cdot\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}=\underline{\frac{8\sqrt{3}}{9}abc}.$$

Anm. Det här talet är en tentamensuppgift från den tid författarna var studenter!

Övning 4.29 (s. 81)



Tältet betraktas som (ett liggande) prisma.

med basen ACD och höjden AE.

Pytagoras' sats ger $AC = 2\sqrt{y^2-z^2}$, så volymen

blir $V = z\cdot\sqrt{y^2-z^2}\cdot x = xz\sqrt{y^2-z^2}$.

Den totala "tältarean" är $2xy + 2xz\sqrt{y^2-z^2}$.

$f(x,y,z) = 2xy + 2z\sqrt{y^2-z^2}$; $g(x,y,z) = xz\sqrt{y^2-z^2} - V = 0$.

$F(x,y,z,\lambda) = 2xy + 2z\sqrt{y^2-z^2} + \lambda(xz\sqrt{y^2-z^2} - V)$

$\frac{\partial F}{\partial x} = 2y + \lambda z\sqrt{y^2-z^2}$, $\frac{\partial F}{\partial y} = 2x + (2+\lambda x)\frac{yz}{\sqrt{y^2-z^2}}$, $\frac{\partial F}{\partial z} = (2+\lambda x)\frac{y^2-2z^2}{\sqrt{y^2-z^2}}$.

$\frac{\partial F}{\partial z} = 0 \Rightarrow y^2 = 2z^2 \Rightarrow \text{Basen} = z\sqrt{y^2-z^2} = z^2 = \frac{V}{x} \Leftrightarrow x = \frac{V}{z^2}$.

$f(x,y,z) = f(\frac{V}{z^2}, \sqrt{2}z, z) = 2z^2 + \frac{2\sqrt{2}V}{z} = \phi(z)$;

$\phi'(z) = 4z - \frac{2\sqrt{2}V}{z^2} = 0 \Leftrightarrow z^3 = \frac{V}{\sqrt{2}} \Leftrightarrow z = \left(\frac{V}{\sqrt{2}}\right)^{1/3}$.

Resultat: Tälthöjden ska vara $\left(\frac{V}{\sqrt{2}}\right)^{1/3} \approx 0{,}89\,V^{1/3}$.

F ovan kallas Lagrangefunktion.

Övning 4.30 (s. 82)

$f(x,y,z) = x+y+z$ (målfunktionen)

$g(x,y,z) = x^2+y^2+z^2-\frac{83}{7} = 0$, $h(x,y,z) = x+2y+3z-4 = 0$

$F(x,y,z,\lambda,\mu) = x+y+z+\lambda(x^2+y^2+z^2-\frac{83}{7}) + \mu(x+2y+3z-4)$.

$\frac{\partial F}{\partial x} = 1+2\lambda x+\mu$, $\frac{\partial F}{\partial y} = 1+2\lambda y+2\mu$, $\frac{\partial F}{\partial z} = 1+2\lambda z+3\mu$;

$\frac{\partial F}{\partial \lambda} = x^2+y^2+z^2-\frac{83}{7}$, $\frac{\partial F}{\partial \mu} = x+2y+3z-4$.

(i) $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0 \Leftrightarrow \lambda = \frac{1}{2(y-2x)} \wedge \mu = \frac{x-y}{y-2x}$; (*)

(ii) $\frac{\partial F}{\partial z} = 0 \Rightarrow 1+2\lambda z+3\mu = 0 \overset{(*)}{\Rightarrow} \frac{x-2y+z}{y-2x} = 0 \Leftrightarrow x = 2y-z$. (**)

(iii) $\frac{\partial F}{\partial \mu} = 0 \overset{(**)}{\Rightarrow} \begin{cases} x-2y+z=0 \\ x+2y+3z=4 \end{cases} \Leftrightarrow \begin{cases} x=-2+4t \\ y=t \\ z=2-2t \end{cases}$, $t\in\mathbb{R}$. (***)

(iv) $\frac{\partial F}{\partial \lambda} = 0 \overset{(***)}{\Rightarrow} x^2+y^2+z^2 = (4t-2)^2+t^2+(2t-2)^2 = \frac{83}{7} \Leftrightarrow$

$\Leftrightarrow 16t^2-16t+4+t^2+4t^2-8t+4 = 21t^2-24t+8 = \frac{83}{7}$

$\Leftrightarrow 21t^2-24t-\frac{27}{7} = 0 \Leftrightarrow t^2-\frac{8}{7}t-\frac{9}{49} = 0 \Leftrightarrow t = \frac{9}{7} \vee$

$\vee\ t = -\frac{1}{7} \overset{(***)}{\Rightarrow} (x,y,z) = (\frac{22}{7}, \frac{9}{7}, -\frac{4}{7}) \vee (x,y,z) = (-\frac{18}{7}, -\frac{1}{7}, \frac{16}{7})$.

$f(\frac{22}{7}, \frac{9}{7}, -\frac{4}{7}) = \frac{27}{7}$, $f(-\frac{18}{7}, -\frac{1}{7}, \frac{16}{7}) = -\frac{3}{7}$.

Resultat: Det största värdet är $\frac{27}{7}$ och det minsta $-\frac{3}{7}$.

Läs nu författarnas lösning på problemet.

Övning 4.31 (s. 82)

$$\begin{cases} f(x,y,z) = x+y+z \\ g(x,y,z) = x^2+y^2+z^2-2=0, \quad h(x,y,z) = x^2+y^2-z=0. \end{cases}$$

Skärningen mellan sfären $x^2+y^2+z^2=2$ och rotationsparaboloiden är cirkeln $(x,y,z)=(\cos t, \sin t, 1)$, $0\le t\le 2\pi$, som är kompakt.

$\phi(t) = f(\cos t, \sin t, 1) = 1+\cos t+\sin t = 1+\sqrt{2}\sin(t+\frac{\pi}{4})$.

$\Rightarrow \quad 1-\sqrt{2} \le f(x,y,z) \le 1+\sqrt{2}$.

Resultat: $\max\{f(x)\} = \underline{1+\sqrt{2}} = f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1)$.

Övning 4.32 (s. 82)

$f(x,y,z) = xy(3-z)$; $g(x,y,z) = x^2+\frac{y^2}{4}+\frac{z^2}{9}-1=0$.

Sfäriska koordinater ger

$x = \sin\theta\cos\phi$, $y = 2\sin\theta\sin\phi$, $z = 3\cos\theta$, $\begin{cases} 0\le\theta\le\frac{\pi}{2} \\ 0\le\phi\le\frac{\pi}{2} \end{cases}$;

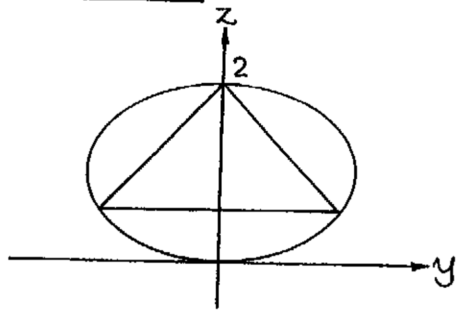
Detta är en parametrisering av ellipsoiden.

$f(\sin\theta\cos\phi, 2\sin\theta\sin\phi, 3\cos\phi) = 2\sin^2\theta\sin\phi\cos\phi \cdot$
$\cdot(3-3\cos\theta) = 3\sin^2\theta(1-\cos\theta)\sin 2\phi = F(\theta,\phi)$.

Som bekant gäller: $0\le\phi\le\frac{\pi}{2} \Rightarrow 0\le\sin 2\phi\le 1$.

Den θ-beroende delen av F har $\max = 3$ för $\theta=\frac{\pi}{2}$, så f:s största värde är $\underline{3}$.

Övning 4.33 (s. 82)



$f(x,y,z) = \frac{1}{3}2x\cdot 2y\cdot(2-z) = \frac{4}{3}xy(2-z)$.

$g(x,y,z) = x^2+\frac{y^2}{4}+(z-1)^2-1=0$

Vi parametriserar ellipsoiden medelst sfäriska koordinater.

$(x,y,z) = (\sin\theta\cos\phi, 2\sin\theta\sin\phi, 1+\cos\theta)$, $\begin{cases} 0\le\theta\le\pi \\ 0\le\phi\le 2\pi \end{cases}$.

$F(\theta,\phi) = f(\sin\theta\cos\phi, 2\sin\theta\sin\phi, 1+\cos\theta) =$
$= \frac{4}{3}\cdot 2\sin^2\theta\sin\phi\cos\phi(1-\cos\theta) =$
$= \frac{4}{3}\sin^2\theta(1-\cos\theta)\sin 2\phi = \psi(\theta)\cdot\chi(\phi)$.

$-1\le\chi(\phi)\le 1$, ty $0\le\phi\le 2\pi$.

$\psi(\theta) = \frac{4}{3}\sin^2\theta(1-\cos\theta) \Rightarrow \psi'(\theta) = \frac{4}{3}\cdot 2\sin\theta\cos\theta(1-\cos\theta) +$

$+\frac{4}{3}\sin^3\theta = \frac{4}{3}\sin\theta\,(2\cos\theta - 2\cos^2\theta + \sin^2\theta) = \frac{4}{3}\sin\theta\,\cdot$

$\cdot\,(1+2\cos\theta - 3\cos^2\theta) = \frac{4}{3}\sin\theta\,(1-\cos\theta)(1+3\cos\theta) = 0 \Leftrightarrow$

$\Leftrightarrow \sin\theta = 0 \vee \cos\theta = 0 \vee \cos\theta = -\frac{1}{3} \Rightarrow \psi(\theta) \equiv 0 \vee$

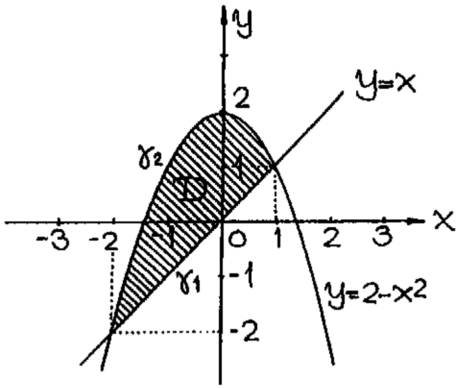
$\vee\ \psi(\theta) = \frac{4}{3}(1-\frac{1}{9})\cdot(1+\frac{1}{3}) = \frac{128}{81}.$

Resultat: Pyramidens volym kan inte bli större än $\frac{128}{81}$ ve.

Blandade problem

Övning 4.34 (s. 82)

$f(x,y) = (x+y+4)e^{x^2+y}$, $D = \{(x,y):\ x \le y \le 2-x^2\}$.



(i) $\mathring{D} = \{(x,y):\ x<y<2-x^2\}$.

$\frac{\partial f}{\partial x} = (1+2x^2+2xy+8x)e^{x^2+y}$, $\frac{\partial f}{\partial y} = (5+x+y)e^{x^2+y}$.

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow x+y+5=0 \wedge 1+2x^2+2xy+8x=0 \Leftrightarrow$

$\Leftrightarrow (x,y) = (\frac{1}{2}, -\frac{11}{2})$ som dock ligger utanför $\mathring{D}$.

(ii) $\gamma_1 = \{(x,x):\ -2 \le x \le 1\}$.

$f(x,x) = 2(x+2)e^{x^2+x} = \phi(x)$, $-2 \le x \le 1$.

$\phi'(x) = 2(2x^2+5x+2)e^{x^2+x} = 0 \Leftrightarrow 2x^2+5x+2=0 \Leftrightarrow x = -\frac{1}{2}$;

$\phi(-2) = 0$, $\phi(-\frac{1}{2}) = 3e^{-1/4}$, $\phi(1) = 6e^2$.

$\gamma_2 = \{(x, 2-x^2):\ -2 \le x \le 1\}$.

$f(x, 2-x^2) = (6+x-x^2)e^2 = \psi(x)$, $-2 \le x \le 1$.

$\psi'(x) = (1-2x)e^2 = 0 \Leftrightarrow x = \frac{1}{2}$;

$\psi(-2) = 0$, $\psi(\frac{1}{2}) = \frac{25}{4}e^2$, $\psi(1) = 6e^2$.

Resultat: Det största värdet är $\frac{25e^2}{4}$ och det minsta 0.

Övning 4.35 (s. 82)

$f(x,y) = xy$, $D = \{(x,y):\ x^2+4y^2 \le 8\}$.

(i) $\mathring{D} = \{(x,y):\ x^2+4y^2 < 8\}$.

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x} \Leftrightarrow y = 0 = x \Leftrightarrow (x,y) = (0,0) \Rightarrow f(0,0) = 0.$

(ii) $\partial D = \{(x,y):\ \frac{x^2}{8} + \frac{y^2}{2} = 1\} = \{(\sqrt{8}\cos\theta, \sqrt{2}\sin\theta):\ 0 \le \theta \le 2\pi\}$.

$f(\sqrt{8}\cos\theta, \sqrt{2}\sin\theta) = 4\sin\theta\cos\theta = 2\sin 2\theta \in [-2, 2].$

Resultat: Största värde $= 2$; minsta värdet $= -2$.

Övning 4.36 (s. 82)

Ellipsoiden är en nivåyta till funktionen

$$f(x,y,z)=x^2+2y^2+3z^2.$$

$\operatorname{grad} f(a,b,c)=(2a,4b,6c)$, $\quad (P=(a,b,c))$.

Planets ekvation är

$\operatorname{grad} f(a,b,c)\cdot(x-a,\,y-b,\,z-c)=0 \Leftrightarrow \underline{ax+2by+3cz=1}$

Jag har utnyttjat det faktum att P ligger på ellipsoiden, dvs. $a^2+2b^2+3c^2=1$.

Skärningspunkterna mellan planet och koordinataxlarna blir

$$A\left(\tfrac{1}{a},0,0\right),\ B\left(0,\tfrac{1}{2b},0\right) \text{ resp. } C\left(0,0,\tfrac{1}{3c}\right).$$

Tetraedervolymen blir $V=\frac{1}{36}\cdot\frac{1}{abc}$.

Vi studerar således problemet

$$\begin{cases} F(a,b,c)=\dfrac{1}{36abc} & \text{(målfunktionen)} \\ G(a,b,c)=a^2+2b^2+3c^2-1=0 & \text{(bivillkor)} \end{cases}$$

Sfäriska (rymdpolära) koordinater ger

$(a,b,c)=(\sin\theta\cos\phi,\ \frac{1}{\sqrt2}\sin\theta\sin\phi,\ \frac{1}{\sqrt3}\cos\phi)$, $\begin{cases} 0\le\theta\le\pi \\ 0\le\phi\le2\pi \end{cases}$

för ellipsoiden. För att förenkla arbetet sätter vi $g(a,b,c)=abc$ på $a^2+2b^2+3c^2=1$.

$g(\sin\theta\cos\phi,\ \frac{1}{\sqrt2}\sin\theta\sin\phi,\ \frac{1}{\sqrt3}\cos\theta)=\frac{1}{2\sqrt6}\sin^2\theta\cos\theta\sin2\phi$.

$\Theta(\theta)=\sin^2\theta\cos\theta \Rightarrow \Theta(\theta)=2\sin\theta-3\sin^3\theta$;

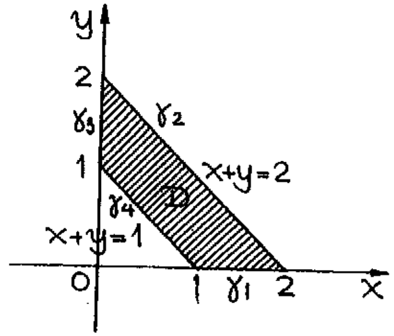
$\Theta'(\theta)=0 \Rightarrow \sin\theta=0 \vee \sin^2\theta=\frac23 \Rightarrow \cos^2\theta=\frac13 \Rightarrow \cos\theta=\frac{1}{\sqrt3}$.

$g(a,b,c)_{max}=\frac{1}{2\sqrt6}\cdot\frac23\cdot\frac{1}{\sqrt3}=\frac{\sqrt2}{18} \Rightarrow F(a,b,c)_{min}=\frac{\sqrt2}{4}$.

<u>Resultat</u>: Den sökta volymen är $\geq\frac{\sqrt2}{4}$ ve.

Övning 4.37 (s. 83)

$f(x,y)=\dfrac{x+y}{1+x^2+y^2}$, $\quad D=\{(x,y):\ 1\le x+y\le 2,\ x,y\ge0\}$.



(i) $\mathring{D}=\{(x,y):\ 1<x+y<2,\ x>0,\ y>0\}$.

$\forall x\in\mathring{D}:\ \dfrac{\partial f}{\partial x}=\dfrac{1-x^2-2xy+y^2}{(1+x^2+y^2)^2} \wedge \dfrac{\partial f}{\partial y}=\dfrac{1+x^2-2xy-y^2}{(1+x^2+y^2)^2}$;

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 1-x^2-2xy+y^2=0 \\ 1+x^2-2xy-y^2=0 \end{cases} \Leftrightarrow \begin{cases} y=x \\ 1-x^2-2xy+y^2=0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow y=x \wedge 2x^2=1 \Leftrightarrow (x,y)=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$; $\ \underline{f\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)=\frac{1}{\sqrt2}}$.

$\gamma_1 = [1,2]\times\{0\} = \{(x,0) : 1\le x\le 2\}$.

$f(x,0) = \dfrac{x}{x^2+1} = \phi(x)$, $1\le x\le 2$.

$\phi'(x) = \dfrac{1-x^2}{(x^2+1)^2} = 0 \Leftrightarrow x = 1$. $\quad \phi(1) = \frac{1}{2}$, $\phi(2) = \frac{2}{5}$.

$\gamma_2 = \{(x,y) : y = 2-x,\ 0\le x\le 2\}$.

$f(x, 2-x) = \dfrac{2}{2x^2-4x+5} = \chi(x)$, $0\le x\le 2$.

$\chi'(x) = -\dfrac{8(x-1)}{(2x^2-4x+5)^2} = 0 \Leftrightarrow x = 1$; $\quad \chi(0) = \frac{2}{5}$, $\chi(2) = \frac{2}{5}$.

$\gamma_3 = \{0\}\times[1,2] = \{(0,y) : 1\le y\le 2\}$.

$f(0,y) = \dfrac{y}{y^2+1} = \psi(y)$, $1\le y\le 2$ (samma som γ_1 ovan).

$\gamma_4 = \{(x, 1-x) : 0\le x\le 1\}$.

$f(x,1-x) = \dfrac{1}{2x^2-2x+2} = \omega(x)$, $0\le x\le 1$.

$\omega'(x) = -\dfrac{2(2x-1)}{(2x^2-2x+2)^2} = 0 \Leftrightarrow x = \frac{1}{2}$; $\quad \omega(0) = \frac{1}{2}$, $\omega(\frac{1}{2}) = \frac{2}{3}$, $\omega(1) = \frac{1}{2}$.

Resultat: $\left\{\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}\right\}^{max}_{min} = \begin{cases} 1/\sqrt{2} \\ 2/5 \end{cases}$.

Övning 4.38 (s. 83)

$f(x,y) = (x+a)^2 + (y+a)^2$ (målfunktionen).

$g(x,y) = x^3 + y^3 - 1 = 0$ (1:a bivillkoret)

$x\ge 0$, $y\ge 0$ (2:a bivillkoret).

$\operatorname{grad} f(x,y) \parallel \operatorname{grad} g(x,y) \Rightarrow (2(x+a), 2(y+a)) \parallel (3x^2, 3y^2)$.

$\Leftrightarrow \dfrac{2(x+a)}{3x^2} = \dfrac{2(y+a)}{3y^2} \Leftrightarrow y^2(x+a) = x^2(y+a) \Leftrightarrow y = x$;

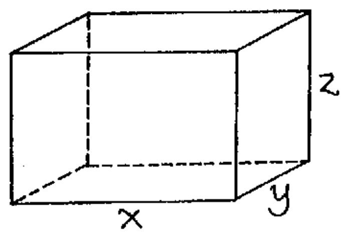
$g(x,x) = 2x^3 - 1 = 0 \Leftrightarrow x = 2^{-1/3} = y$;

$x\ge 0 \wedge y\ge 0 \Rightarrow 0\le x\le 1 \wedge 0\le y\le 1$ (ty $y = \sqrt[3]{1-x^3}$).

Kurvan är symmetrisk m.a.p. linjen $y = x$, s.a. de punkter som är intressanta är $(0,1)$, $(1,0)$ och $(2^{-1/3}, 2^{-1/3})$. $f(0,1) = f(1,0) = 2a^2 + 2a + 1$ och $f(2^{-1/3}, 2^{-1/3})$ $= 2a^2 + 2^{5/3}a + 2^{1/3}$.

Resultat: Det största avståndet är $\sqrt{2a^2 + 2^{5/3}a + 2^{1/3}}$ det minsta $\sqrt{2a^2+2a+1}$.

Övning 4.39 (s. 83)



(m)

$f(x,y,z) = xyz$, $\quad g(x,y,z) = 2(xy + yz + xz - 1) = 0$.

$\operatorname{grad} f(x) \parallel \operatorname{grad} g(x) \Rightarrow (yz, xz, xy) \parallel (y+z, x+z, x+y) \Leftrightarrow$

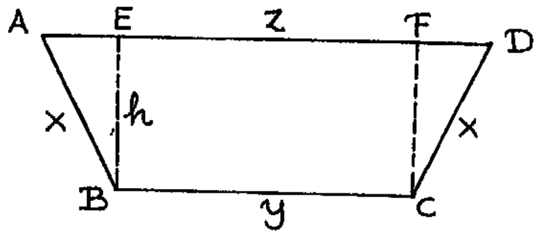
$\Leftrightarrow \dfrac{yz}{2(y+z)} = \dfrac{xz}{2(x+z)} = \dfrac{xy}{2(x+y)} \Leftrightarrow x = y = z \Rightarrow g(x,x,x) = 6x^2-2=0$

$\Rightarrow x=y=z=\frac{1}{\sqrt{3}} \Rightarrow f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\frac{\sqrt{3}}{9}$.

<u>Resultat</u>: Den största volymen är $\frac{\sqrt{3}}{9}$ ve.

Övning 4.40 (s. 83)

(cm)



AD=z

$AE=FD=\frac{z-y}{2} \Rightarrow h=BE=CF=\sqrt{x^2-\left(\frac{z-y}{2}\right)^2}$;

$A=\frac{AD+BC}{2}\cdot BE=\frac{1}{2}(y+z)\sqrt{x^2-\left(\frac{z-y}{2}\right)^2}$

$$\begin{cases} f(x,y,z)=\frac{1}{2}(y+z)\sqrt{x^2-\left(\frac{z-y}{2}\right)^2} & \text{(målfunktionen)} \\ g(x,y,z)=2x+y-60=0 & \text{(bivillkor)} \end{cases}$$

$F(x,y,z,\lambda)=\frac{1}{2}(y+z)\sqrt{x^2-\left(\frac{z-y}{2}\right)^2}+\lambda(2x+y-60)$;

$$\begin{cases} \frac{\partial F}{\partial x}=\frac{x(y+z)}{\sqrt{4x^2-(y-z)^2}}+2\lambda ; \\ \frac{\partial F}{\partial y}=\frac{z^2-y^2}{4\sqrt{4x^2-(y-z)^2}}+\frac{1}{4}\sqrt{4x^2-(y-z)^2}+\lambda ; \\ \frac{\partial F}{\partial z}=\frac{y^2-z^2}{4\sqrt{4x^2-(y-z)^2}}+\frac{1}{4}\sqrt{4x^2-(y-z)^2} ; \end{cases}$$

$$\begin{cases} \frac{\partial F}{\partial y}=\frac{\partial F}{\partial z}=0 \Rightarrow 2\lambda=\frac{y^2-z^2}{\sqrt{4x^2-(y-z)^2}} \\ \frac{\partial F}{\partial x}=0 \Rightarrow 2\lambda=-\frac{x(y+z)}{\sqrt{4x^2-(y-z)^2}} \end{cases} \Rightarrow y^2-z^2=-x(y+z) \Leftrightarrow$$

$\Leftrightarrow x=z-y \Rightarrow h=\frac{\sqrt{3}}{2}x$ (efter lite algebra).

$2x+y=60 \Leftrightarrow y=60-2x$.

ΔABC är en halv liksidig (triangel), dvs.

$AE=\frac{1}{2}x \Rightarrow z=y+x=(60-2x)+x=60-x$.

$f(x,y,z)=(x,20-x,60-x)=\dots=\frac{3\sqrt{3}}{4}(40x-x^2)=\phi(x)$.

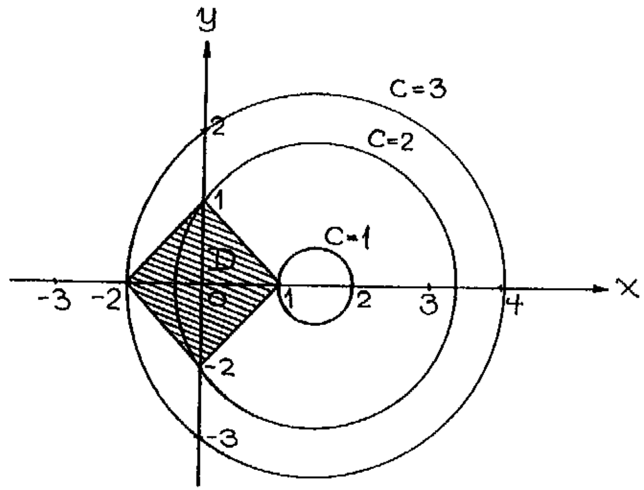
$\phi'(x)=\frac{3\sqrt{3}}{2}(20-x)=0 \Leftrightarrow x=20 \Rightarrow \phi(20)=\underline{300\sqrt{3}}$.

<u>Resultat</u>: Arean kan bli maximalt $300\sqrt{3}\,cm^2$.

Övning 4.41 (s. 83)

$f(x,y)=\left(x-\frac{3}{2}\right)^2+y^2-\frac{9}{4}$, $D=\{(x,y): |x|+|y|\leq 1\}$.

Nivåkurvorna till $f(x)$ är cirklar med centrum i punkten $\left(\frac{3}{2},0\right)$.

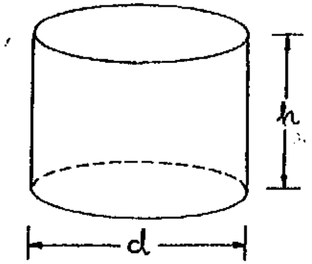


$f(1,0)=-2$, $f(0,1)=f(0,-1)=1$, $f(-1,0)=-4$.

forts.

Resultat: $\max\limits_{x\in D} f(x)=4$, $\min\limits_{x\in D}\{f(x)\}=-2$

Övning 4.42 (s.83)



Volymen $V=\pi r^2 h=\pi(\frac{d}{2})^2 h=\frac{1}{4}\pi d^2 h$.

Arean $A=2\cdot\pi(\frac{d}{2})^2+\pi dh=\frac{1}{2}\pi d^2+\pi dh$.

$V=1 \Rightarrow \pi d^2 h=4$;

$f(d,h)=\frac{1}{2}\pi d^2+\pi dh$, $g(d,h)=\pi d^2 h-4=0$.

$\operatorname{grad} f(d,h) \parallel \operatorname{grad} g(d,h) \Rightarrow \frac{\cancel{\pi}(d+h)}{2\cancel{\pi}dh}=\frac{\cancel{\pi}d}{\cancel{\pi}d^2} \Leftrightarrow$

$\Leftrightarrow \frac{d+h}{2dh}=\frac{1}{d} \Leftrightarrow \frac{1}{h}+\frac{1}{d}=\frac{2}{d} \Leftrightarrow \frac{1}{h}=\frac{1}{d} \Leftrightarrow \frac{d}{h}=1$.

Resultat: Det sökta förhållandet är 1.

Övning 4.43 (s.83)

$f(x,y,z)=x+y+z$; $g(x,y)=\sqrt{x^2+y^2-1}-z=0$, $1\le|x|^2\le 3$.

$f(x,y,\sqrt{x^2+y^2-1})=x+y+\sqrt{x^2+y^2-1}=F(x,y)$, $1\le|x|^2\le 3$.

$F(r\cos\theta, r\sin\theta)=r\cos\theta+r\sin\theta+\sqrt{r^2-1}=G(r,\theta)$;

$G(r,\theta)=\sqrt{2}r\sin(\theta+\frac{\pi}{4})+\sqrt{r^2-1}$; $D=[1,\sqrt{3}]\times[0,2\pi]$.

(i) $\mathring{D}=]1,\sqrt{3}[\times]0,2\pi[$.

$\frac{\partial G}{\partial r}=\sqrt{2}\sin(\theta+\frac{\pi}{4})+\frac{r}{\sqrt{r^2-1}}$, $\frac{\partial G}{\partial\theta}=\sqrt{2}r\cos(\theta+\frac{\pi}{4})$;

$\frac{\partial G}{\partial\theta}=0 \Rightarrow \cos(\theta+\frac{\pi}{4})=0 \Leftrightarrow \theta+\frac{\pi}{4}=\frac{\pi}{2} \vee \theta+\frac{\pi}{4}=\frac{3\pi}{2} \Leftrightarrow$

$\Leftrightarrow \theta=\frac{\pi}{4} \vee \theta=\frac{5\pi}{4}$ (*)

$\frac{\partial G}{\partial r}=0 \overset{(*)}{\Rightarrow} -\sqrt{2}+\frac{r}{\sqrt{r^2-1}} \Leftrightarrow \frac{r^2}{r^2-1}=2 \Leftrightarrow r^2=2r^2-2 \Leftrightarrow r=\sqrt{2}$.

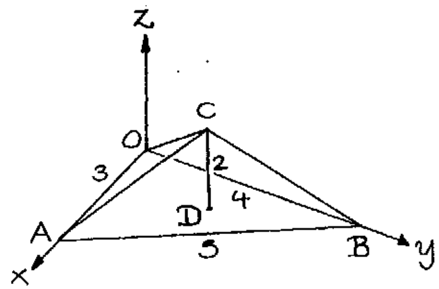
$G(\sqrt{2},\frac{5\pi}{4})=1-\sqrt{2}$.

(ii) $r=1 \Rightarrow G(1,\theta)=\sqrt{2}\sin(\theta+\frac{\pi}{4})\in[-\sqrt{2},\sqrt{2}]$.

$r=\sqrt{3} \Rightarrow G(\sqrt{3},\theta)=\sqrt{6}\sin(\theta+\frac{\pi}{4})+\sqrt{2} \in[\sqrt{2}-\sqrt{6},\sqrt{2}+\sqrt{6}]$.

Resultat: $-\sqrt{2}\le x+y+z\le\sqrt{2}+\sqrt{6} \Leftarrow 1\le|x|\le\sqrt{3}$.

Övning 4.44 (s.83)



$O=(0,0,0)$, $A=(3,0,0)$, $B=(0,4,0)$, $C=(x,y,2)$, $D=(x,y,0)$.

(i) ΔOAC: $\overline{OA}=(3,0,0)$, $\overline{OC}=(x,y,2)$;

forts.

$$S_1 = \frac{1}{2}\overline{OA}\times\overline{OC} = \frac{1}{2}\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 3 & 0 & 0 \\ x & y & 2 \end{vmatrix} = \frac{1}{2}(0,-6,3y);$$

<u>ΔOBC</u>: $\overline{OB} = (0,4,0)$, $\overline{OC} = (x,y,2)$;

$$S_2 = \frac{1}{2}\overline{OC}\times\overline{OB} = \frac{1}{2}\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ x & y & 2 \\ 0 & 4 & 0 \end{vmatrix} = (-4,0,2x);$$

<u>ΔABC</u>: $\overline{AC} = \overline{OC}-\overline{OA} = (x-3,y,2)$, $\overline{AB} = \overline{OB}-\overline{OA} = (-3,4,0)$.

$$S_3 = \frac{1}{2}\overline{AB}\times\overline{AC} = \frac{1}{2}\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -3 & 4 & 0 \\ x-3 & y & 2 \end{vmatrix} = \frac{1}{2}(8,6,-3y-4x+12).$$

$$S_1^2+S_2^2+S_3^2 = \frac{1}{4}(36+9y^2)+16+4x^2+\frac{1}{4}(64+36+9y^2+16x^2$$
$$+144+24xy-72y-96x) = 8x^2+\frac{9}{2}y^2+6xy-24x-18y+86.$$

(ii) $\underline{f(x,y) = 8x^2+\frac{9}{2}y^2+6xy-24x-18y+86}$;

$\underline{D = \{(x,y):\ 4x+3y\le 12,\ x\ge 0, y\ge 0\}}$.

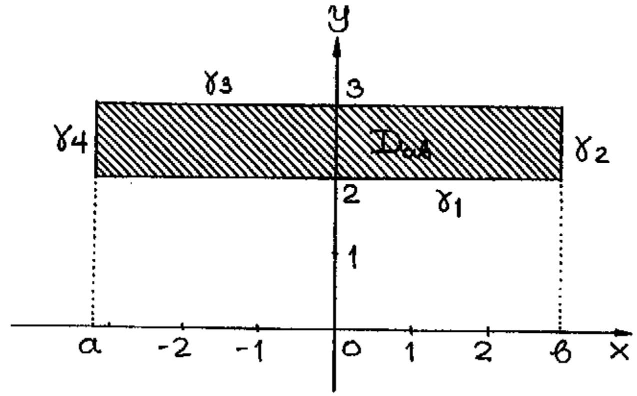
$\frac{\partial f}{\partial x} = 16x+6y-24$, $\frac{\partial f}{\partial y} = 9y+6x-18$;

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y} \Rightarrow \begin{cases} 8x+3y=12 \\ 2x+3y=6 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=\frac{4}{3} \end{cases} \Rightarrow \underline{(x,y,z) = (1,\tfrac{4}{3},0)}.$$

<u>Övning 4.45 (s. 83)</u>

$\underline{f(x,y) = (x^2+y)e^{x-y},\quad D = \{(x,y):\ 2\le y\le 3\}}$.

D är uppenbarligen icke-kompakt.



(i) $\underline{\mathring{D}_{ab} = \{(x,y):\ a<x<b,\ 2<y<3\} =]a,b[\times]1,3[}$.

$\frac{\partial f}{\partial x} = (2x+x^2+y)e^{x-y}$, $\frac{\partial f}{\partial y} = (1-x^2-y)e^{x-y}$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 2x+x^2+y=0 \\ 1-x^2-y=0 \end{cases} \Leftrightarrow \begin{cases} y=1-x^2 \\ 2x+1=0 \end{cases} \Leftrightarrow \begin{cases} x=-\frac{1}{2} \\ y=\frac{3}{4} \end{cases};$$

$(x,y) = (-\frac{1}{2},\frac{3}{4}) \notin \mathring{D} \Leftrightarrow$ inga stationära på "bandet".

(ii) $\underline{\gamma_1 = \{(x,2):\ a\le x\le b\} = [a,b]\times\{2\}}$.

$f(x,2) = (x^2+2)e^{x-2} = \phi(x)$, $a\le x\le b$.

$\phi'(x) = (x^2+2x+2)e^{x-2} > 0 \Rightarrow \phi$ växande och >0.

$\lim_{a\to-\infty}\phi(a) = 0$, $\lim_{b\to\infty}\phi(b) = \infty$.

$\underline{\gamma_2 = \{b\}\times[2,3] = \{(b,y):\ 2\le y\le 3\}}$.

$f(b,y) = (b^2+y)e^{b-y} = \psi(y)$, $2\le y\le 3$;

$\psi'(y) = (1-y-b^2)e^{b-y} = 0 \Leftrightarrow y = 1-b^2 < 0$ för stora b.

$\psi(\ ) = (b^2+2)e^{b-2} \xrightarrow[b\to\infty]{} \infty$, $\psi(3) = (b^2+3)e^{b-3} \xrightarrow[b\to\infty]{} \infty$.

$\gamma_3 = [a,b]\times\{3\} = \{(x,3):\ a\le x\le b\}$.

$f(x,3) = (x^2+3)e^{x-3} = \chi(x)$, $a\le x\le b$.

$\chi'(x) = (x^2+2x+3)e^{x-3} > 0 \Rightarrow \chi$ växande och positiv.

$\lim_{b\to\infty}\chi(b)=\infty$, $\lim_{a\to-\infty}\chi(a)=0^+$

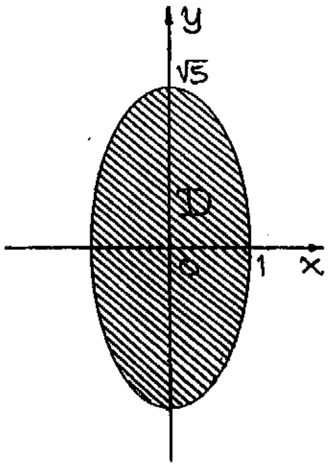
P.s.s. γ_2 behandlas γ_4 (den ger inget nytt).

Jag har visat att $V_f = \mathbb{R}_+$, så $f(x,y)=e^{-1}$ är en nivåkurva (eg. båge) i det smala bandet.

Resultat: Det finns säkert punkter (x,y) s.a. $f(x,y)=\frac{1}{e}$.

Övning 4.46 (s. 83)

$f(x,y) = 10+x^2+xy$, $D=\{(x,y):\ x^2+\frac{y^2}{5}\le 1\}$.



forts.

grad $f(x,y) = (2x+y,\ x) \Rightarrow |\nabla f(x)|^2 = 5x^2+4xy+y^2$;

$F(x,y)=5x^2+4xy+y^2$, $D=\{(x,y):\ x^2+\frac{y^2}{5}\le 1\}$.

(i) $\forall x\in \mathring{D}$: $\frac{\partial F}{\partial x} = 10x+4y \wedge \frac{\partial F}{\partial y}=4x+2y$.

$\frac{\partial F}{\partial x}=0=\frac{\partial F}{\partial y} \Rightarrow (x,y)=(0,0)$; $f(0,0)=0$.

(ii) $\partial D = \{(x,y):\ x^2+\frac{y^2}{5}=1\} = \{(\cos\theta,\ \sqrt5\sin\theta):\ 0\le\theta\le 2\pi\}$

$f(\cos\theta,\sqrt5\sin\theta) = 5+2\sqrt5\sin 2\theta \in [5-2\sqrt5,\ 5+2\sqrt5]$.

f_{max} antas för $\sin 2\theta=1 \Leftrightarrow \theta = \frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$.

$\theta=\frac{\pi}{4} \Rightarrow (x,y)=(\frac{\sqrt2}{2},\frac{\sqrt{10}}{2}) \wedge v=(\sqrt2+\frac{\sqrt{10}}{2},\frac{\sqrt2}{2})$;

$\theta=\frac{3\pi}{4} \Rightarrow (x,y)=(-\frac{\sqrt2}{2},\frac{\sqrt{10}}{2}) \wedge v=(-\sqrt2+\frac{\sqrt{10}}{2},-\frac{\sqrt2}{2})$;

$\theta=\frac{5\pi}{4} \Rightarrow (x,y)=(-\frac{\sqrt2}{2},-\frac{\sqrt{10}}{2}) \wedge v=(\sqrt2-\frac{\sqrt{10}}{2},\frac{\sqrt2}{2})$;

$\theta=\frac{7\pi}{4} \Rightarrow (x,y)=(\frac{\sqrt2}{2},-\frac{\sqrt{10}}{2}) \wedge v=(-\sqrt2-\frac{\sqrt{10}}{2},-\frac{\sqrt2}{2})$.

Resultat: Den största lutningen är $\sqrt{5+2\sqrt5}$. Denna antas i ovanstående punkter och riktningar.

Anm. Studer Sats 7 (s. 67) om tolkningen på gradienten och speciellt dess belopp.

5. Några användningar av differentialkalkyl

Derivation under integraltecknet

Övning 5.1 (s. 102)

a) $f(s,x)=\cos(sx) \Rightarrow \frac{\partial f}{\partial s}=-x\cdot\sin(sx) \Rightarrow f, f'_s \in C^1$;

$$\begin{cases} F'(s)=\frac{d}{ds}\int_0^1 \cos(sx)dx=\int_0^1\left(\frac{\partial}{\partial s}\cos(sx)\right)dx=-\int_0^1 x\sin(sx)dx. \\ \frac{d}{ds}\frac{\sin(s)}{s}=-\frac{\sin(s)}{s^2}+\frac{\cos(s)}{s} \end{cases}$$

b) $F'(s)=-\int_0^1 x\cdot\sin(sx)\,dx=\frac{\cos(s)}{s}-\frac{\sin(s)}{s^2}$;

$F'(1)=-\int_0^1 x\sin x\,dx=\cos 1-\sin 1$;

$\int_0^1 x\sin x\,dx=\sin 1-\cos 1.$

Övning 5.2 (s. 102)

$F(s)=\int_0^1 e^{sx}dx=\frac{1}{s}[e^{sx}]_0^1=\frac{e^s-1}{s}$;

$F'(s)=\frac{d}{ds}\int_0^1 e^{sx}dx=\int_0^1\frac{\partial}{\partial s}e^{sx}dx=\int_0^1 xe^{sx}dx=\frac{se^s-e^s+1}{s^2}$;

$F''(s)=\frac{d}{ds}\int_0^1 xe^{sx}dx=\int_0^1\frac{\partial}{\partial s}xe^{sx}dx=\int_0^1 x^2e^{sx}dx=$

$=\frac{s^2(\cancel{e^s}+se^s-\cancel{e^s})-2s(se^s-e^s+1)}{s^4}=\frac{s^2e^s-2se^s+2e^s-2}{s^3}$; (*)

$F''(1)=\int_0^1 x^2e^x dx \overset{(*)}{=} e-2.$

Övning 5.3 (s. 102)

a) $F(s)=\left[\int_0^s e^{-x^2}dx\right]^2 \Rightarrow F'(s)=2\int_0^s e^{-x^2}dx\,\frac{d}{ds}\int_0^s e^{-x^2}dx=$

$=2\int_0^s e^{-x^2}dx\cdot e^{-s^2}=2e^{-s^2}\int_0^s e^{-x^2}dx$;

$G(s)=\int_0^1\frac{e^{-s^2(x^2+1)}}{x^2+1}dx \Rightarrow G'(s)=\frac{d}{ds}\int_0^s\frac{e^{-s^2(x^2+1)}}{x^2+1}dx=$

$=\int_0^1\frac{\partial}{\partial s}\frac{e^{-s^2(x^2+1)}}{x^2+1}dx=\int_0^1\frac{e^{-s^2(x^2+1)}}{x^2+1}\frac{\partial}{\partial s}(-s^2(x^2+1))dx=$

$=\int_0^1\frac{e^{-s^2(x^2+1)}}{x^2+1}(-2s(x^2+1))dx=-\int_0^1 2se^{-s^2(x^2+1)}dx=$

$=-2e^{-s^2}\int_0^1 e^{-(sx)^2}d(sx)\,[u=sx]=-2e^{-s^2}\int_0^1 e^{-u^2}du=$

$=-F'(s) \Leftrightarrow G'(s)+F'(s)=0$ (V.S.V.)

b) $F(0)+G(0)=\left[\int_0^0 e^{-x^2}dx\right]^2+\int_0^1\frac{1}{x^2+1}dx=0+\arctan 1=\frac{\pi}{4}.$

c) $F'(s)+G'(s)=(F(s)+G(s))'=0 \Leftrightarrow F(s)+G(s)=$ konstant $=$

$=F(0)+G(0)=\frac{\pi}{4}.$

d) $F(s)+G(s)=\frac{\pi}{4} \Leftrightarrow G(s)=\frac{\pi}{4}-F(s) \Rightarrow \lim_{s\to\infty}G(s)=\pi/4-$

$-\lim_{s\to\infty}\left(\int_0^s e^{-x^2}dx\right)^2$; (*)

$\forall(x,s)\in\mathbb{R}^2:\ 0\leq\frac{e^{-s^2(x^2+1)}}{x^2+1}\leq\frac{1}{x^2+1} \Rightarrow F(s)<\frac{\pi}{4}$;

$\lim_{s\to\infty}\int_0^1\frac{e^{-s^2(x^2+1)}}{x^2+1}dx=\int_0^1\lim_{s\to\infty}\frac{e^{-s^2(x^2+1)}}{x^2+1}dx=0.$

(*) $\Leftrightarrow 0=\frac{\pi}{4}-\left(\int_0^\infty e^{-x^2}dx\right)^2 \Leftrightarrow \int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$

Övning 5.4 (s. 102)

$s>0 \Rightarrow \frac{1}{s}>0 \Rightarrow f(s,x)=\frac{\sin(sx)}{x}\in C^1 \Rightarrow F(s)\in C^2.$

$F(s)=\int_{1/s}^s\frac{\sin(sx)}{x}dx \Rightarrow F'(s)=\frac{d}{ds}\int_{1/s}^s\frac{\sin(sx)}{x}dx=$

$$= \int_{1/s}^{s} \frac{\partial}{\partial s}\frac{\sin(sx)}{x}\,dx + \frac{\sin(s^2)}{s} - \frac{\sin 1}{1/s}\left(-\frac{1}{s^2}\right) = \int_{1/s}^{s} \cos(sx)\,dx + \frac{\sin(s^2)}{s} + \frac{\sin 1}{s} = \left[\frac{\sin(sx)}{s}\right]_{1/s}^{s} + \frac{\sin s^2}{s} + \frac{\sin 1}{s} = \frac{\sin s^2}{s} - \frac{\sin 1}{s} + \frac{\sin(s^2)}{s} + \frac{\sin 1}{s} = 2\cdot\frac{\sin s^2}{s}.$$

Övning 5.5 (s. 103)

$$F(s) = \int_0^s e^{-(s-x)}\cos x^2\,dx \Rightarrow F'(s) = \frac{d}{ds}\int_0^s e^{-(s-x)}\cos x^2\,dx =$$

$$= \int_0^s \frac{\partial}{\partial s} e^{-(s-x)}\cos x^2\,dx + \cos(s^2) = -\int_0^s e^{-(s-x)}\cos x^2\,dx + \cos(s^2) = -F(s) + \cos(s^2) \Leftrightarrow \underline{F'(s)+F(s) = \cos(s^2)}.$$

$$F(s) = \int_0^s e^{-(s-x)}\cos x^2\,dx \Rightarrow F(0) = \int_0^0 e^{-(s-x)}\cos x^2\,dx = 0.$$

Övning 5.6 (s. 103)

$$F(x) = \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy \Rightarrow F'(x) = \frac{d}{dx}\int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy$$

$$= \int_0^x \frac{\partial}{\partial x}\frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy = \int_0^x \frac{\cancel{(n-1)}(x-y)^{n-2}}{(n-2)!\cancel{(n-1)}} f(y)\,dy =$$

$$= \int_0^x \frac{(x-y)^{n-2}}{(n-2)!} f(y)\,dy \Rightarrow F''(x) = \frac{d}{dx}\int_0^x \frac{(x-y)^{n-2}}{(n-2)!} f(y)\,dy =$$

$$= \int_0^x \frac{\partial}{\partial x}\frac{(x-y)^{n-2}}{(n-2)!} f(y)\,dy = \int_0^x \frac{\cancel{(n-2)}(x-y)^{n-3}}{(n-3)!\cancel{(n-2)}} f(y)\,dy =$$

$$= \int_0^x \frac{(x-y)^{n-3}}{(n-3)!} f(y)\,dy \Rightarrow \dots \Rightarrow F^{(n-1)}(x) = \frac{d^{n-1}}{dx^{n-1}}\int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy$$

$$= \int_0^x \frac{\partial^{n-1}}{\partial x^{n-1}}\frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy = \int_0^x f(y)\,dy \Rightarrow F^{(n)}(x) = f(x).$$

Övning 5.7 (s. 103)

Se nästa sida.

a) $F(s) = \int_0^\infty e^{-x^2}\cos(sx)\,dx \Rightarrow F'(s) = \frac{d}{ds}\int_0^\infty e^{-x^2}\cos(sx)\,dx =$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-x^2}\cos(sx)\,dx = -\int_0^\infty x e^{-x^2}\sin(sx)\,dx =$$

$$= \int_0^\infty e^{-x^2}\sin(sx)\,\tfrac{1}{2}d(-x^2) = \tfrac{1}{2}\left[\cancel{e^{-x^2}\sin(sx)}^{\,0}\right]_0^\infty -$$

$$-\frac{1}{2}\int_0^\infty e^{-x^2}\cos(sx)\,s\,dx = -\frac{s}{2}\int_0^\infty e^{-x^2}\cos(sx)\,dx = -\frac{s}{2}F(s)$$

$$\Leftrightarrow F'(s) + \frac{s}{2}F(s) = 0. \quad (*)$$

b) $\phi(s) = \frac{s}{2} \Rightarrow \Phi(s) = \int \phi(s)\,ds = \frac{s^2}{4} \Rightarrow \underline{\mu(s) = e^{s^2/4}}$;

$$e^{s^2/4}F'(s) + \frac{s}{2}e^{s^2/4}F(s) = 0 \Leftrightarrow \frac{d}{ds}\,e^{s^2/4}F(s) = 0 \Leftrightarrow$$

$$\Leftrightarrow e^{s^2/4}F(s) = C = F(0) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \Leftrightarrow \underline{F(s) = \frac{\sqrt{\pi}}{2}e^{-s^2/4}}.$$

Anm. $\mu(s) = e^{s^2/4}$ är en s.k. integrerande faktor till differentialekvationen (*).

Övning 5.8 (s. 103)

$$F(s) = \int_0^\infty e^{-(x^2+s^2/x^2)}\,dx \Rightarrow F'(s) = \frac{d}{ds}\int_0^\infty e^{-(x^2+s^2/x^2)}\,dx =$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-(x^2+s^2/x^2)}\,dx = \int_0^\infty \left(-\frac{2s}{x^2}\right)e^{-(x^2+s^2/x^2)}\,dx \left[y = \frac{s}{x}\right] =$$

$$= 2\int_\infty^0 e^{-(s^2/y^2+y^2)}\,dy = -2\int_0^\infty e^{-(x^2+s^2/x^2)}\,dx = -2F(s) \Leftrightarrow$$

$$\Leftrightarrow F'(s) + 2F(s) = 0.$$

Övning 5.9 (s. 103)

a) $F(s) = \int_0^\infty \frac{e^{-sx}-e^{-2x}}{x}\,dx \Rightarrow F'(s) = \frac{d}{ds}\int_0^\infty \frac{e^{-sx}-e^{-2x}}{x}\,dx$

$$=\int_0^\infty \frac{\partial}{\partial s}\frac{e^{-sx}-e^{-2x}}{x}dx=\int_0^\infty(-x)\cdot\frac{e^{-sx}}{x}dx=-\int_0^\infty e^{-sx}dx=-\frac{1}{s}.$$

b) $F(2)=\lim_{s\to 2}\int_0^\infty \lim_{s\to 2}\frac{e^{-sx}-e^{-2x}}{x}dx=0$;

$$F'(s)=-\frac{1}{s}\Rightarrow F(s)-F(2)=-\int_2^s\frac{1}{s}ds=\ln\frac{2}{s}\Leftrightarrow \underline{F(s)=\ln\frac{2}{s}}.$$

Derivator inom termodynamiken

Övning 5.10 (s. 104)

$U=f(T)\Rightarrow C_V=\left(\frac{\partial U}{\partial T}\right)_V=f'(T).$ (*)

$H=U+pV=f(T)+RT\Rightarrow C_p=\left(\frac{\partial H}{\partial T}\right)_p=f'(t)+R\overset{(*)}{=}C_V+R.$

Övning 5.11 (S.104)

T konstant $\Rightarrow pV=RT=$ konstant $\Rightarrow H=f(T)+pV=$

$=$ konstant $\Rightarrow\left(\frac{\partial H}{\partial V}\right)_T=0.$

Övning 5.12 (S.104)

$U=f(T)\Rightarrow dU=f'(T)dT=C_VdT.$

$dH=\frac{\partial H}{\partial T}dT+\frac{\partial H}{\partial p}dp=(C_V+R)dT=C_pdT.$

Övning 5.13 (s.104)

$H=U+pV\Rightarrow dH=dU+d(pV)=dU+pdV+Vdp=$

$=(dU+pdV)+Vdp=0+Vdp=Vdp.$

Övning 5.14 (s. 104)

a) $\begin{cases} dU+pdV=0\\ dH=Vdp\end{cases}\Rightarrow\begin{cases}C_VdT+pdV=0\\ C_pdT=Vdp\end{cases}\Leftrightarrow\begin{cases}C_VdT=-pdV\\ C_pdT=Vdp\end{cases}\Rightarrow$

$\Rightarrow\frac{C_V\cancel{dT}}{C_p\cancel{dT}}=-\frac{p}{V}\frac{dV}{dp}\Leftrightarrow\frac{C_V}{C_p}+\frac{p}{V}\frac{dV}{dp}=0\Leftrightarrow\frac{dp}{p}+\frac{C_p}{C_V}\frac{dV}{V}=0$; (*)

b) $\gamma=\frac{C_p}{C_V}\overset{(*)}{\Rightarrow}\frac{dp}{p}+\gamma\frac{dV}{V}=0\Leftrightarrow\ln p+\gamma\cdot\ln V=\ln C\Leftrightarrow$

$\Leftrightarrow\ln p+\ln V^\gamma=\ln C\Leftrightarrow\ln pV^\gamma=\ln C\Leftrightarrow pV^\gamma=C.$

Blandade problem

Övning 5.15 (s.104)

$F(s)=\int_0^\infty\frac{\arctan(sx)}{(x^2+1)\cdot x}dx\Rightarrow f(s,x)=\frac{\arctan(sx)}{(x^2+1)x}$,

$\frac{\partial f}{\partial s}=\frac{\cancel{x}}{\cancel{x}(x^2+1)(s^2x^2+1)}\Rightarrow\left|\frac{\partial f}{\partial s}\right|=\frac{1}{(x^2+1)(s^2x^2+1)}\leq\frac{1}{x^2+1}=g(x).$

(Se Sats 3, s. 167).

$\int_0^\infty\frac{1}{x^2+1}dx=\lim_{R\to\infty}[\arctan x]_0^R=\lim_{R\to\infty}\arctan R=\frac{\pi}{2}.$

$F(s)=\int_0^\infty\frac{\arctan(sx)}{(x^2+1)x}dx\Rightarrow F'(s)=\frac{d}{ds}\int_0^\infty\frac{\arctan(sx)}{x(x^2+1)}dx=$

$=\int_0^\infty\frac{\partial}{\partial s}\frac{\arctan(sx)}{x(x^2+1)}dx=\int_0^\infty\frac{1}{(x^2+1)(s^2x^2+1)}dx$;

$$f'_s(s,x)=\frac{1}{(x^2+1)(s^2x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{s^2x^2+1}=$$

$$=\frac{(Ax+B)(s^2x^2+1)+(Cx+D)(x^2+1)}{(x^2+1)(s^2x^2+1)}=$$

$$=\frac{As^2x^3+Ax+Bs^2x^2+B+Cx^3+Cx+Dx^2+D}{(x^2+1)(s^2x^2+1)}=$$

$$= \frac{(As^2+C)x^3+(Bs^2+D)x^2+(A+C)x+B+D}{(x^2+1)(s^2x^2+1)} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} As^2+C=0\\ Bs^2+D=0 \\ A+C=0 \\ B+D=1\end{cases} \Leftrightarrow \begin{cases} A=0\\ C=0\\ D=1-B\\ Bs^2+1-B=0\end{cases} \Leftrightarrow \begin{cases}A=0\\ B=1/(1-s^2)\\ C=0\\ D=-s^2/(1-s^2)\end{cases}$$

$$F'(s) = \int_0^\infty \left(\frac{1}{1-s^2}\cdot\frac{1}{x^2+1} - \frac{s^2}{1-s^2}\cdot\frac{1}{s^2x^2+1}\right)dx = -\frac{1}{s^2-1}\int_0^\infty \frac{1}{x^2+1}dx +$$

$$+\frac{s^2}{s^2-1}\int_0^\infty \frac{1}{s^2x^2+1}dx = -\frac{\pi}{2}\,\frac{1}{s^2-1} + \frac{s^2}{s^2-1}\int_0^\infty \frac{1}{(sx)^2+1}dx\ [u=sx] =$$

$$= -\frac{\pi}{2}\,\frac{1}{s^2-1} + \frac{s}{s^2-1}\int_0^\infty \frac{1}{u^2+1}du = -\frac{\pi}{2}\,\frac{1}{s^2-1} + \frac{\pi}{2}\,\frac{s}{s^2-1} = \frac{\pi}{2}\,\frac{1}{s+1}$$

$$\Rightarrow F(s)-F(0) = \frac{\pi}{2}\int_0^s \frac{1}{\sigma+1}d\sigma = \frac{\pi}{2}\ln(1+s) \Leftrightarrow \underline{F(s) = \frac{\pi}{2}\ln(1+s)}.$$

Övning 5.16) (s.105)

a) $F(s) = \int_0^{\pi/2} \ln(\sin^2x + s^2\cos^2x)\,dx$;

$$F'(s) = \frac{d}{ds}\int_0^{\pi/2} \ln(\sin^2x+s^2\cos^2x)\,dx =$$

$$= \int_0^{\pi/2} \frac{\partial}{\partial s}\ln(\sin^2x+s^2\cos^2x)\,dx$$

$$= \int_0^{\pi/2} \frac{2s\cdot\cos^2x}{\sin^2x+s^2\cdot\cos^2x}dx =$$

$$= \int_0^{\pi/2} \frac{2s}{\tan^2x+s^2}dx \left[\begin{array}{l} u=\tan x \Rightarrow du = \sec^2x\,dx \\ x=0\Rightarrow u=0;\ x=\frac{\pi}{2}\Rightarrow u=\infty\end{array}\right] =$$

$$= \int_0^\infty \frac{2s}{(u^2+s^2)(u^2+1)}du =$$

$$= \int_0^\infty \frac{2}{s^2-1}\left(\frac{s}{u^2+1} - \frac{s}{u^2+s^2}\right)du =$$

$$= \frac{2}{s^2-1}\left[s\arctan u - \arctan\frac{u}{s}\right]_0^\infty =$$

$$= \frac{2}{s^2-1}\cdot\frac{\pi}{2}(s-1) = \pi/(s+1).$$

forts.

b) $F(1) = \int_0^{\pi/2} \ln(\cos^2x+\sin^2x)\,dx = \int_0^{\pi/2}\ln 1\,dx = 0.$

$$F(2) - \cancel{F(1)}^{\,0} = \int_1^2 \frac{\pi}{s+1}ds = \pi\left[\ln(1+s)\right]_1^2 = \pi\ln\frac{3}{2}.$$

Övning 5.17 (s. 105)

$$\begin{cases} dS = \frac{1}{T}dU + \frac{p}{T}dV \Leftrightarrow dU = TdS - pdV \\ U = U(S,V) \Rightarrow dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV \end{cases} \Rightarrow \begin{cases} \left(\frac{\partial U}{\partial S}\right)_V = T \\ \left(\frac{\partial U}{\partial V}\right)_S = -p \end{cases} \quad (*)$$

$$U\in C^2 \Rightarrow \frac{\partial^2U}{\partial V\partial S} = \frac{\partial^2U}{\partial S\partial V} \Rightarrow \left(\frac{\partial}{\partial V}\right)_S\left(\frac{\partial U}{\partial S}\right)_V = \left(\frac{\partial}{\partial S}\right)_V\left(\frac{\partial U}{\partial V}\right)_S \Rightarrow$$

$$\Rightarrow \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V.$$

Övning 5.18 (105)

$F(s) = \int_0^\infty \frac{\sin x}{x}e^{-sx}dx$ (laplacetransformen till $\frac{\sin x}{x}$). (*)

a) $$F'(s) = \frac{d}{ds}\int_0^\infty \frac{\sin x}{x}e^{-sx}dx = \int_0^\infty \frac{\sin x}{x}\frac{\partial}{\partial s}e^{-sx}dx =$$

$$= \int_0^\infty -\sin x\cdot e^{-sx}dx = \lim_{R\to\infty}\left(-\int_0^R \sin x\, e^{-sx}dx\right) =$$

$$= \lim_{R\to\infty}\left[\frac{e^{-sx}(s\cdot\sin x+\cos x)}{s^2+1}\right]_0^R = -\frac{1}{s^2+1},\ (s>0) \Rightarrow$$

$$\Rightarrow F(s) = C - \arctan s; \quad (**)$$

b) $$|F(s)| = \left|\int_0^\infty \frac{\sin x}{x}e^{-sx}dx\right| \le \int_0^\infty \frac{|\sin x|}{x}e^{-sx}dx \le$$

$$\le \int_0^\infty e^{-sx}dx = 1/s;$$

c) $$F\in C^1 \overset{(*)}{\Rightarrow} \lim_{s\to\infty}F(s) = 0 \overset{(**)}{\Rightarrow} 0 = C - \frac{\pi}{2} \Leftrightarrow C = \frac{\pi}{2} \Rightarrow$$

$$\Rightarrow F(s) = \operatorname{arccot} s = \frac{\pi}{2} - \arctan s \Rightarrow F(0) = \frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}dx$$

Övning 5.19 (s. 105)

$$u(x)=\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt \Rightarrow |u(x)|=\left|\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt\right| \le$$

$$\le \int_{-1}^{1}\frac{|\cos(xt)|}{\sqrt{1-t^2}}\,dt \le \int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt = 2\arcsin 1=\pi.$$

Enligt Sats 3 kan vi derivera under integraltecknet.

$$u'(x)=\frac{d}{dx}\int_{-1}^{1}\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt=\int_{-1}^{1}\frac{\partial}{\partial x}\,\frac{\cos(xt)}{\sqrt{1-t^2}}\,dt=-\int_{-1}^{1}\frac{t\cdot\sin(xt)}{\sqrt{1-t^2}}\,dt$$

$$u''(x)=\frac{d}{dx}\int_{-1}^{1}-\frac{t\sin xt}{\sqrt{1-t^2}}\,dt=\int_{-1}^{1}\frac{\partial}{\partial x}\left(-\frac{t\cdot\sin xt}{\sqrt{1-t^2}}\right)dt=-\int_{-1}^{1}\frac{t^2\cos xt}{\sqrt{1-t^2}}\,dt;$$

$$VL=u''(x)+\frac{1}{x}u'(x)+u(x)=(u''(x)+u(x))+\frac{1}{x}u'(x)=$$

$$=\int_{-1}^{1}\frac{(1-t^2)\cos xt}{\sqrt{1-t^2}}\,dt-\frac{1}{x}\int_{-1}^{1}\frac{t}{\sqrt{1-t^2}}\sin(xt)\,dt=(\text{P.I.})=$$

$$=\int_{-1}^{1}\sqrt{1-t^2}\cos xt\,dt+\underbrace{\frac{1}{x}\left[\sqrt{1-t^2}\sin xt\right]_{-1}^{1}}_{0}-\int_{-1}^{1}\sqrt{1-t^2}\cos xt\,dt=0$$

Övning 5.20 (s. 105)

$$V(tn_1,tn_2,\dots,tn_m)=t\cdot V(n_1,n_2,\dots,n_m)$$

Vi deriverar ledvis m.a.p. t och får

$$\frac{d}{dt}V(tn_1,tn_2,\dots,tn_m)=V(n_1,n_2,\dots,n_m).\Leftrightarrow$$

$$\Leftrightarrow\left(\frac{\partial V}{\partial n_1}\right)_{n_{j+1}}n_1+\left(\frac{\partial V}{\partial n_2}\right)_{n_{j+2}}n_2+\dots+\left(\frac{\partial V}{\partial n_m}\right)_{n_{j+m}}n_n=$$

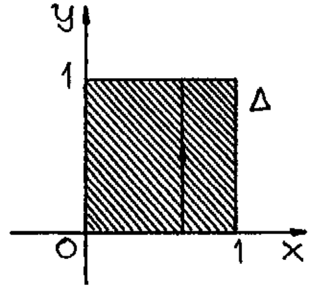
$$=V(n_1,n_2,\dots,n_m)\Leftrightarrow V_1n_1+V_2n_2+\dots+V_mn_m=V.$$

Detta är Eulers sats om homogena funktioner.

6 Dubbelintegraler

Dubbelintegral över rektangel

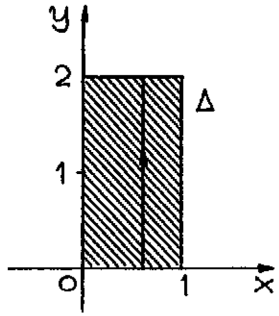
Övning 6.1 (s. 113)



$$\iint_\Delta(xy+y^2)\,dxdy=\int_0^1\left(\int_0^1(xy+y^2)\,dy\right)dx=\int_0^1\left(\left[\tfrac12xy^2+\tfrac13y^3\right]_0^1\right)dx$$

$$=\int_0^1\left(\tfrac12x+\tfrac13\right)dx=\left[\tfrac14x^2+\tfrac13x\right]_0^1=\tfrac14+\tfrac13=\tfrac{7}{12}.$$

Övning 6.2 (s. 113)

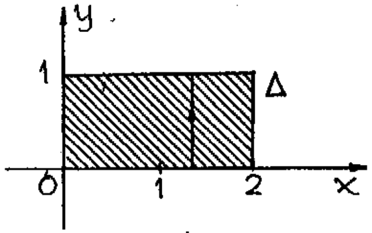


$$\iint_\Delta\frac{1}{1+x+y}\,dxdy=\int_0^1\left(\int_0^2\frac{1}{1+x+y}\,dy\right)dx=\int_0^1\left(\left[\ln(1+x+y)\right]_0^2\right)dx$$

$$=\int_0^1(\ln(3+x)-\ln(1+x))\,dx=\left[(x+3)\ln(x+3)-(x+1)\ln(x+1)\right]_0^1$$

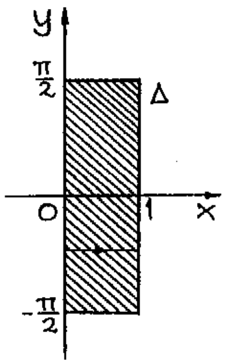
$$=4\ln 4-2\ln 2-3\ln 3=6\ln 2-3\ln 3=3\ln\frac43.$$

Övning 6.3 (s. 113)



$$\iint_\Delta x e^{xy}\,dx\,dy = \int_0^2 \left(\int_0^1 x e^{xy}\,dy \right) dx = \int_0^2 \left([e^{xy}]_0^1 \right) dx$$
$$= \int_0^2 (e^x - 1)\,dx = [e^x - x] = e^2 - 2 - 1 = \underline{e^2 - 3}.$$

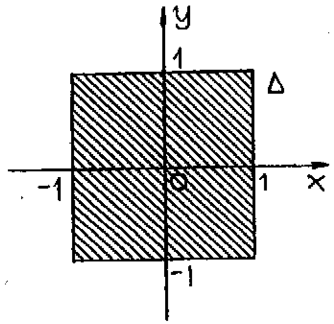
Övning 6.4 (s. 113)



$$\iint_\Delta y \cdot \sin(y + xy)\,dx\,dy = \int_{-\pi/2}^{\pi/2} \left(\int_0^1 y \sin(y+xy)\,dx \right) dy =$$
$$= \int_{-\pi/2}^{\pi/2} \left([-\cos(y+xy)]_0^1 \right) dy = \int_{-\pi/2}^{\pi/2} (\cos y - \cos 2y)\,dy =$$
$$= \left[\sin y - \tfrac{1}{2}\sin 2y\right]_{-\pi/2}^{\pi/2} = \sin\tfrac{\pi}{2} - \tfrac{1}{2}\sin\pi + \sin\tfrac{\pi}{2} - \tfrac{1}{2}\sin\pi =$$
$$= 2\sin\tfrac{\pi}{2} = \underline{2}.$$

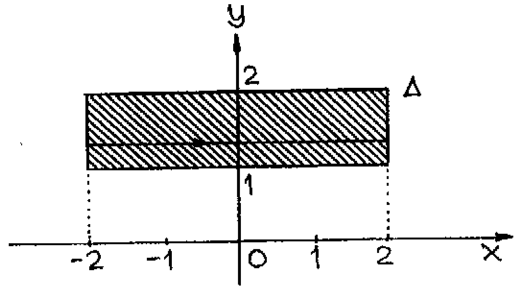
Övning 6.5 (s. 113)

Se nästa sida.



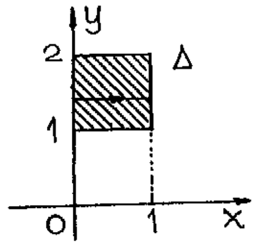
$$\iint_\Delta x e^{-(x^2+y^2)}\,dx\,dy = \underbrace{\int_{-1}^{1} x e^{-x^2}\,dx}_{0} \cdot \int_{-1}^{1} e^{-y^2}\,dy = 0$$

Övning 6.6 (s. 113)



$$f(x,y) = \frac{y \cdot \sin x}{(1+x^2+y^2)^3} \Rightarrow f(-x,y) = -f(x,y) \quad (\text{udda i } x)$$
$$\iint_\Delta f(x,y)\,dx\,dy = \int_1^2 \left(\int_{-2}^{2} f(x,y)\,dx \right) dy = 0.$$

Övning 6.7 (s. 113)



forts.

$$\iint_\Delta e^{xy}(1+xy)\,dxdy = \int_1^2 \Big(\underbrace{\int_0^1 e^{xy}(1+xy)\,dx}_{A(y)}\Big)dy;$$

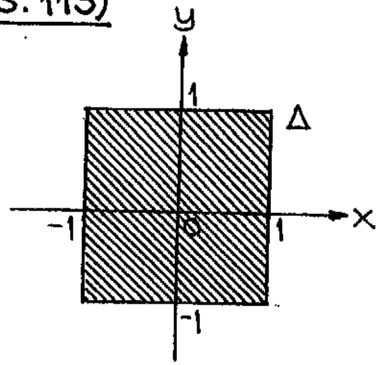
$$A(y)=\int_0^1 e^{xy}(1+xy)\,dx = \left[\frac{1}{y}e^{xy}(1+xy)\right]_{x=0}^{x=1} -$$

$$-\int_0^1 e^{xy}\,dx = \frac{1}{y}e^{y}(1+y)-\frac{1}{y}-\left[\frac{1}{y}e^{xy}\right]_{x=0}^{x=1} =$$

$$=\frac{1}{y}e^{y}+e^{y}-\frac{1}{y}-\left(\frac{1}{y}e^{y}-\frac{1}{y}\right)=e^{y};$$

$$\int_1^2 e^{y}\,dy = [e^{y}]_1^2 = \underline{e^2-e}.$$

Övning 6.8 (s. 113)



$$\iint_\Delta xye^{x+y}\,dxdy = \int_{-1}^1 xe^{x}\,dx\int_{-1}^1 ye^{y}\,dy = \left(\left[(x-1)e^{x}\right]_{-1}^{1}\right)^2 =$$

$$=(-2e^{-1})^2 = 4e^{-2}.$$

Övning 6.9 (s. 113)

$$\iint_\Delta xy\sin(x+y)\,dxdy = \iint_\Delta xy(\sin x\cos y+\cos x\sin y)\,dxdy =$$

$$=\iint_\Delta xy\sin x\cos y\,dxdy + \iint_\Delta xy\cos x\sin y\,dxdy =$$

$$=\int_0^\pi x\sin x\,dx\int_0^\pi y\cos y\,dy + \int_0^\pi x\cos x\,dx\int_0^\pi y\sin y\,dy =$$

$$=2\int_0^\pi x\sin x\,dx\int_0^\pi x\cos x\,dx = 2\cdot I_1\cdot I_2;$$

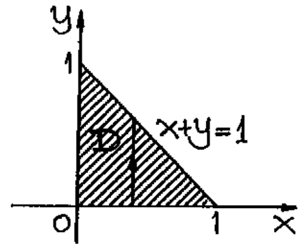
$$I_1=\int_0^\pi x\sin x\,dx = [-x\cos x]_0^\pi + \int_0^\pi \cos x\,dx = \pi.$$

$$I_2=\int_0^\pi x\cos x\,dx = [x\sin x]_0^\pi - \int_0^\pi \sin x\,dx = -2.$$

Resultat: $\displaystyle\iint_{[0,\pi]^2} xy\sin(x+y)\,dxdy = -4\pi.$

Integration över godtyckliga områden

Övning 6.10 (s. 113)

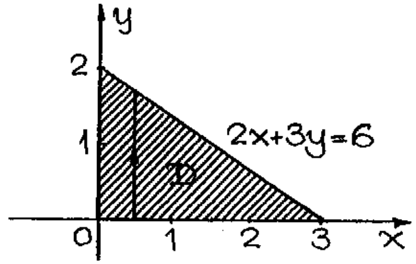


$$\iint_D \frac{y}{1+x}\,dxdy = \int_0^1\left(\int_0^{1-x}\frac{y}{x+1}\,dy\right)dx = \int_0^1\frac{1}{x+1}\left(\int_0^{1-x} y\,dy\right)dx =$$

$$=\int_0^1 \frac{1}{x+1}\left(\left[\frac{1}{2}y^2\right]_0^{1-x}\right)dx = \frac{1}{2}\int_0^1\frac{(1-x)^2}{x+1}\,dx = \frac{1}{2}\int_0^1\left(x-3+\frac{4}{x+1}\right)dx$$

$$=\frac{1}{2}\left[\frac{1}{2}x^2-3x+4\ln(x+1)\right]_0^1 = \frac{1}{2}\left(\frac{1}{2}-3+4\ln 2\right) = \underline{2\ln 2-\frac{5}{4}}.$$

Övning 6.11 (s. 114)



forts.

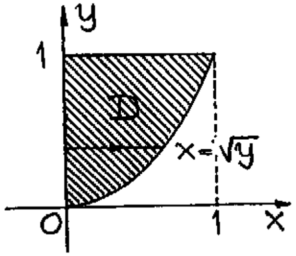
$$2x+3y=6 \Leftrightarrow \frac{2}{3}x+y=2 \Leftrightarrow y=2-\frac{2}{3}x.$$

$$\iint_D e^{-2x-3y}\,dxdy = \int_0^3\left(\int_0^{2-2x/3} e^{-2x-3y}\,dy\right)dx =$$

$$=\int_0^3\left(\left[-\frac{1}{3}e^{-2x-3y}\right]_{y=0}^{y=-2x/3+2}\right)dx = \frac{1}{3}\int_0^3\left(e^{-2x}-e^{-6}\right)dx =$$

$$=\frac{1}{3}\left[-\frac{1}{2}e^{-2x}-e^{-6}x\right]_0^3 = \frac{1}{3}\left(-\frac{1}{2}e^{-6}-3e^{-6}+\frac{1}{2}\right)=\underline{\frac{1}{6}\left(1-7e^{-6}\right)}.$$

Övning 6.12 (S. 114)



$$\iint_D \frac{y}{x^2+1}\,dxdy = \int_0^1\left(\int_0^{\sqrt{y}}\frac{x}{y^2+1}\,dx\right)dy = \int_0^1 \frac{1}{2}\left(\left[\frac{x^2}{y^2+1}\right]_0^{\sqrt{y}}\right)dy =$$

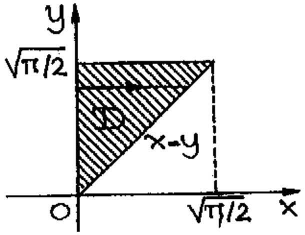
$$=\frac{1}{2}\int_0^1\frac{y}{y^2+1}\,dy = \frac{1}{4}\left[\ln(y^2+1)\right]_0^1 = \underline{\frac{1}{4}\ln 2}.$$

Övning 6.13 (S. 114)

a) $\displaystyle\int_0^{\sqrt{\pi/2}}\left(\int_0^y \cos y^2\,dx\right)dy = \int_0^{\sqrt{\pi/2}}\cos y^2\left([x]_0^y\right)dy =$

$$=\int_0^{\sqrt{\pi/2}} y\cos y^2\,dy = \frac{1}{2}\left[\sin y^2\right]_0^{\sqrt{\pi/2}} = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}.$$

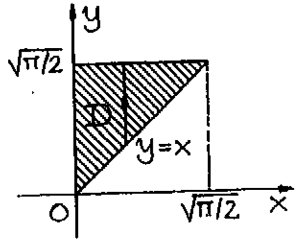
b)



forts.

$$\iint_D \cos y^2\,dxdy = \int_0^{\sqrt{\pi/2}}\cos y^2\int_0^y dx = \frac{1}{2} \text{ (enl. a)).}$$

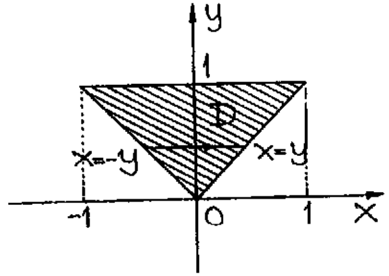
c)



$$\iint_D \cos y^2\,dxdy = \int_0^{\sqrt{\pi/2}}\Big(\underbrace{\int_x^{\sqrt{\pi/2}}\cos y^2\,dy}_{A(x)}\Big)dx.$$

$A(x)$ är omöjligt att genomföra exakt (med elementära funktioner). Resultatet blir i alla fall detsamma, dvs. $\frac{1}{2}$. Det gäller att välja rätt (integrations)väg!

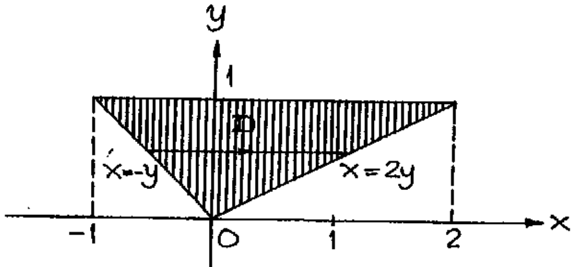
Övning 6.14 (S. 114)



$$\iint_D e^{-y^2}\,dxdy = \int_0^1\left(\int_{-y}^{y} e^{-y^2}\,dx\right)dy = \int_0^1 e^{-y^2}[x]_{-y}^{y})\,dy =$$

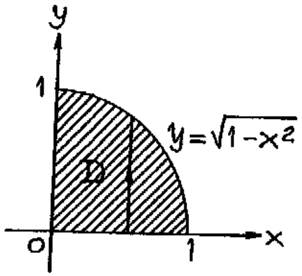
$$=\int_0^1 2ye^{-y^2}\,dy = \left[-e^{-y^2}\right]_0^1 = \underline{1-e^{-1}}.$$

Övning 6.15 (s. 114)



$$\iint_D \frac{1}{1+(x-2y)^2}dxdy = \int_0^1 dy\int_{-y}^{2y}dx\,\frac{1}{1+(x-2y)^2} =$$
$$=\int_0^1 d\ \left[\arctan(x-2y)\right]_{-y}^{2y} = \int_0^1 \arctan 3y\,dy\ [u=3y] =$$
$$=\frac13\int_0^3 \arctan u\,du = \frac13\left[u\arctan u\right]_0^3 - \frac13\int_0^3\frac{u}{u^2+1}du =$$
$$=\arctan 3 - \frac16\left[\ln(u^2+1)\right]_0^3 = \arctan 3 - \frac16\ln 10.$$

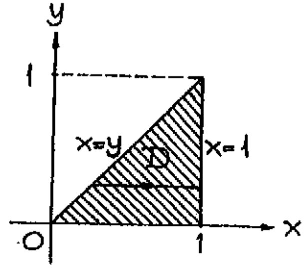
Övning 6.16 (s. 114)



$$\iint_D\frac{xy}{(1+y^2)^2}dxdy = \int_0^1\left(\int_0^{\sqrt{1-x^2}}\frac{xy}{(1+y^2)^2}dy\right)dx = \int_0^1 A(x)dx;$$
$$A(x) = \int_0^{\sqrt{1-x^2}}\frac{xy}{(1+y^2)^2}dy = -\frac12 x\left[\frac{1}{y^2+1}\right]_0^{\sqrt{1-x^2}} = \frac12\left(x-\frac{x}{2-x^2}\right)\Rightarrow$$
$$\Rightarrow \int_0^1 A(x)dx = \frac14\int_0^1\left(2x+\frac{-2x}{2-x^2}\right)dx = \frac14\left[x^2+\ln(2-x^2)\right]_0^1 =$$
$$=\frac14(1+\ln 1-\ln 2) = \frac{1-\ln 2}{4}.$$

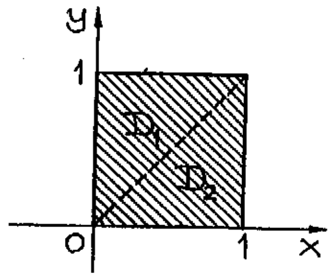
Övning 6.17 (s. 114)

a)



$$\iint_D xy\sqrt{x^2+y^2}\,dxdy = \int_0^1 y\left(\int_y^1 x\sqrt{x^2+y^2}\,dx\right)dy =$$
$$=\int_0^1 y\left(\frac12\int_y^1 2x\sqrt{x^2+y^2}\,dx\right)dy = \int_0^1\left(\frac13 y\left[(x^2+y^2)^{3/2}\right]_y^1\right)dy =$$
$$=\frac13\int_0^1\left(y(y^2+1)^{3/2}-2^{3/2}y^4\right)dy = \frac13\left[\frac15(y^2+1)^{5/2}-\frac{2^{3/2}}{5}y^5\right]_0^1 =$$
$$=\frac{1}{15}(4\sqrt2-2\sqrt2-1) = \frac{2\sqrt2-1}{15}.$$

b)



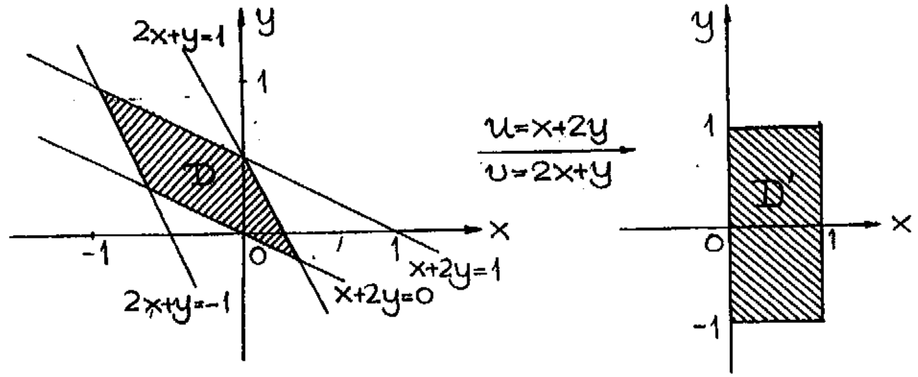
$D = D_1\cup D_2$

$$\iint_D f(x,y)\,dxdy = \iint_{D_1\cup D_2} f(x,y)\,dxdy = \iint_{D_1} f(x,y)\,dxdy +$$
$$+\iint_{D_2} f(x,y)\,dxdy = \int_0^1\left(\int_x^1 y\sqrt{x^2+y^2}\,dy\right)x\,dx +$$
$$+\int_0^1\left(\int_y^1 x\sqrt{x^2+y^2}\,dx\right)y\,dy = 2\int_0^1\left(\int_y^1 x\sqrt{x^2+y^2}\,dx\right)y\,dy =$$

$= (\text{Se } a)) = \frac{2}{15}(2\sqrt{2}-1).$

Variabelbyte i dubbelintegraler

Övning 6.18 (s.115)



$D=\{(x,y): 0\le x+2y\le 1, -1\le 2x+y\le 1\} \longrightarrow D'=[0,1]\times[-1,1].$

$$\begin{cases} u=x+2y \\ v=2x+y \end{cases} \Leftrightarrow \begin{cases} x=-\frac{1}{3}u+\frac{2}{3}v \\ y=\frac{2}{3}u-\frac{1}{3}v \end{cases} \Rightarrow \frac{d(x,y)}{d(u,v)}=\begin{vmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

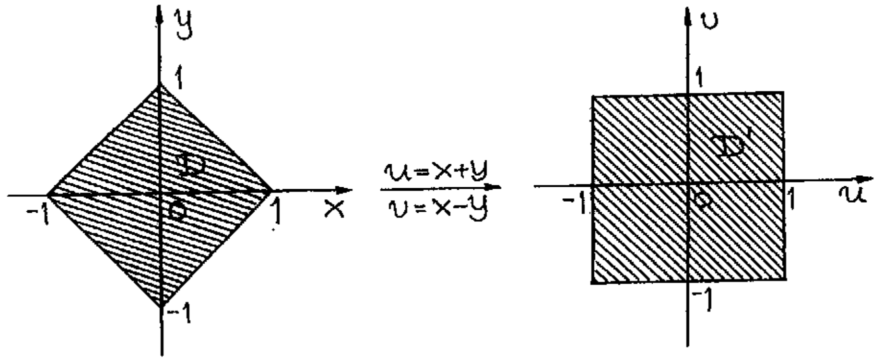
$$\iint_D (x+2y)\cos(2x+y)\,dxdy \begin{bmatrix} u=x+2y \\ v=2x+y \end{bmatrix} = \frac{1}{3}\iint_{D'} u\cos v\,dudv =$$

$$=\frac{1}{3}\int_0^1 u\,du \int_{-1}^1 \cos v\,dv = \frac{1}{3}\left[\frac{1}{2}u^2\right]_0^1 \cdot [\sin v]_{-1}^1 = \frac{1}{3}\sin 1.$$

Övning 6.19 (s.115)

$f(x,y)=(x^2-y^2)^2=(x+y)^{10}(x-y)^{10};$

$D=\{(x,y): |x|+|y|\le 1\}=\{*: -1\le x+y\le 1\}\cap\{*: -1\le x-y\le 1\}.$

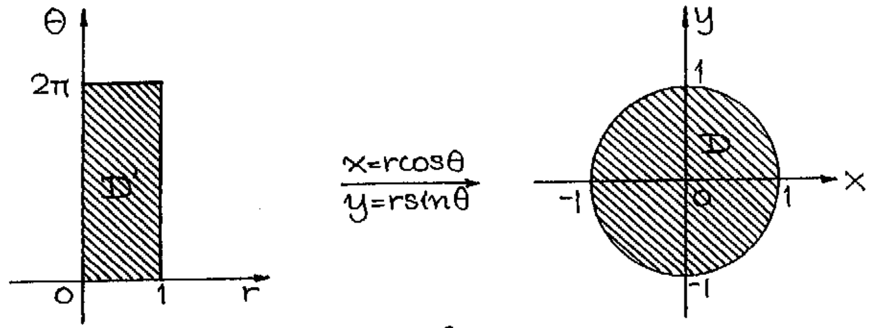


$$\begin{cases} u=x+y \\ v=x-y \end{cases} \Leftrightarrow \begin{cases} x=\frac{1}{2}u+\frac{1}{2}v \\ y=\frac{1}{2}u-\frac{1}{2}v \end{cases} \Rightarrow \frac{d(x,y)}{d(u,v)}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2};$$

$$\iint_D (x^2-y^2)^{10}\,dxdy \begin{bmatrix} u=x+y \\ v=x-y \end{bmatrix} = \iint_{D'} (uv)^{10}\cdot\frac{1}{2}\,dudv =$$

$$=\frac{1}{2}\int_{-1}^1 u^{10}\,du \int_{-1}^1 v^{10}\,dv = \frac{1}{2}\left[\frac{u^{11}}{11}\right]_{-1}^1\cdot\left[\frac{v^{11}}{11}\right]_{-1}^1 = \frac{1}{2}\left(\frac{2}{11}\right)^2 = \frac{2}{121}.$$

Övning 6.20 (s.115)

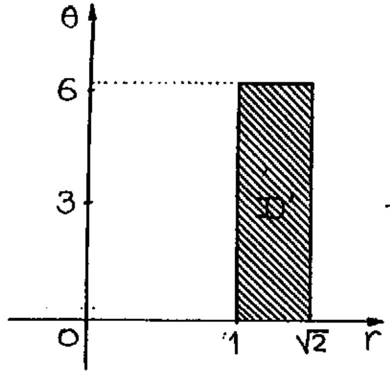


$$\iint_D \frac{(x+y)^2}{1+x^2+y^2}\,dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \iint_{D'} \frac{r^2(1+\sin 2\theta)}{r^2+1}\,r\,drd\theta =$$

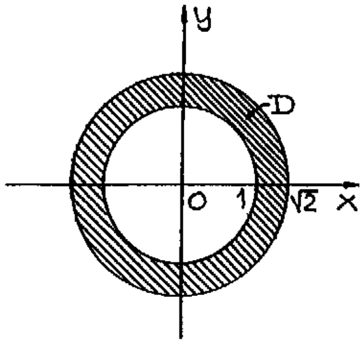
$$=\iint_{D'} \frac{r^3}{r^2+1}(1+\sin 2\theta)\,drd\theta = \int_0^1 \left(r-\frac{r}{r^2+1}\right)dr \int_0^{2\pi}(1+\sin 2\theta)\,d\theta$$

$$=\frac{1}{2}\left[r^2-\ln(r^2+1)\right]_0^1\cdot\left[\theta-\frac{1}{2}\cos 2\theta\right]_0^{2\pi} = \dots = \pi(1-\ln 2).$$

Övning 6.21 (s. 115)



$\dfrac{x=r\cos\theta}{y=r\sin\theta}$

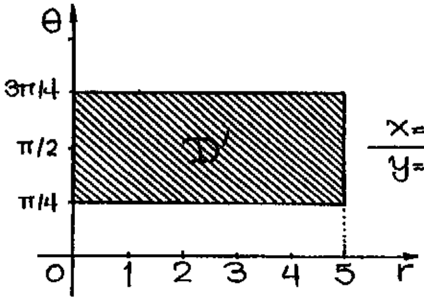


$$\iint_D \ln(1+x^2+y^2)\,dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \iint_{D'} \ln(1+r^2)\cdot r\,dr\,d\theta =$$

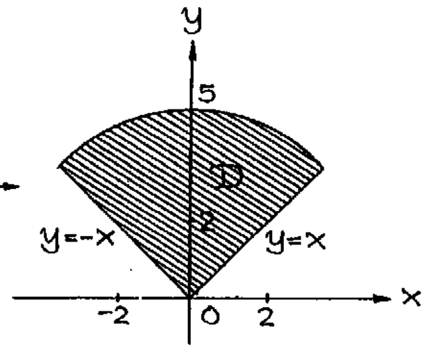
$$= \int_1^{\sqrt2} r\ln(1+r^2)\,dr \cdot \int_0^{2\pi} d\theta = \pi\int_1^{\sqrt2} 2r\ln(1+r^2)\,dr\,[t=r^2+1]$$

$$= \pi\int_2^3 \ln t\,dt = \pi\left[t\ln t - t\right]_2^3 = \underline{\pi(3\ln3-2\ln2-1)}.$$

Övning 6.22 (s. 115)



$\dfrac{x=r\cos\theta}{y=r\sin\theta}$



$D=\{(x,y):\ x^2+y^2\le 25,\ y\ge|x|\}.$

$D'=\{(r,\theta):\ 0\le r\le 5,\ \frac{\pi}{4}\le\theta\le\frac{3\pi}{4}\}.$

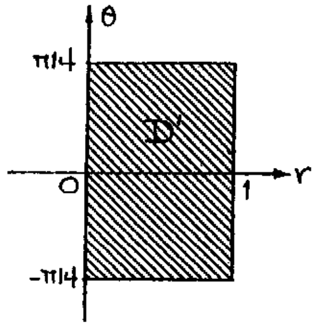
$$\iint_D x^2e^{x^2+y^2}\,dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \iint_{D'} r^2\cos^2\theta\, e^{r^2}\, r\,dr\,d\theta =$$

$$= \int_0^5 r^3e^{r^2}dr\int_{\pi/4}^{3\pi/4}\frac12(1+\cos2\theta)\,d\theta = \frac12\int_0^5 r^3e^{r^2}dr\left[\theta+\frac{\sin2\theta}{2}\right]_{\pi/4}^{3\pi/4}$$

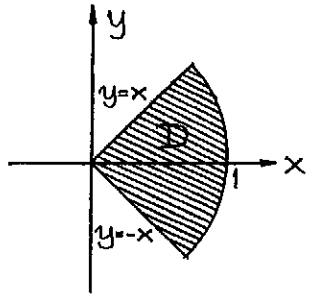
$$= \frac12\left(\frac{\pi}{2}-1\right)\int_0^5 r^3e^{r^2}dr \left[\begin{array}{l|l} u=r^2 & r=0\Rightarrow u=0 \\ du=2r\,dr & r=5\Rightarrow u=25 \end{array}\right] =$$

$$= \frac{\pi-2}{8}\int_0^{25} ue^u\,du = \frac{\pi-2}{8}\left[(u-1)e^u\right]_0^{25} = \underline{\frac{\pi-2}{8}(24e^{25}+1)}.$$

Övning 6.23 (s. 115)



$\dfrac{x=r\cos\theta}{y=r\sin\theta}$



$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \Rightarrow \begin{cases} x^2-y^2=r^2\cos2\theta \\ 2xy=r^2\sin2\theta \end{cases} \wedge \frac{d(x,y)}{d(r,\theta)}=r.$$

$$\iint_D (x^2-y^2)e^{2xy}\,dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \iint_{D'} r^2\cos2\theta\, e^{r^2\sin2\theta}\, r\,dr\,d\theta$$

$$= \int_0^1 dr\, r^3\int_{-\pi/4}^{\pi/4}\cos2\theta\, e^{r^2\sin2\theta}\,d\theta = \int_0^1 r^3 I(r)\,dr\,;$$

$$I(r) = \int_{-\pi/4}^{\pi/4}\cos2\theta\, e^{r^2\sin2\theta}\,d\theta \begin{bmatrix} u=r^2\sin2\theta \\ du=2r^2\cos2\theta\,d\theta \end{bmatrix} =$$

$$= \frac{1}{2r^2}\int_{-r^2}^{r^2} e^u\,du = \frac{1}{r^2}\cdot\frac{e^{r^2}-e^{-r^2}}{2} = \frac{\sinh r^2}{r^2}\,;$$

$$\int_0^1 r^3\cdot\frac{1}{r^2}\sinh r^2\,dr = \int_0^r r\sinh r^2\,dr \begin{bmatrix} v=r^2 \\ dv=2r\,dr \end{bmatrix} =$$

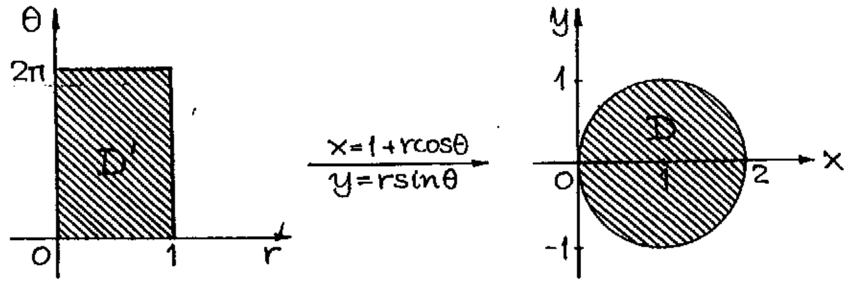
$$= \int_0^1 \frac12\sinh v\,dv = \frac12\left[\cosh v\right]_0^1 = \frac{\cosh1-1}{2}$$

<u>Resultat</u>: $\iint_D (x^2-y^2)e^{2xy}\,dxdy = \dfrac{e+e^{-1}-2}{4}.$

Övning 6.24 (s. 115)

$x^2+y^2-2x \equiv (x-1)^2+y^2-1 \Rightarrow \underline{D=\{(x,y):\ (x-1)^2+y^2\le 1\}}$

$\begin{cases} x=1+r\cos\theta \\ y=r\sin\theta \end{cases} \Rightarrow x^2+y^2=1+2r\cos\theta+r^2\ ;\ \begin{cases} 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{cases}$



$\iint_D (x^2+y^2)\,dxdy \left[\begin{matrix} x=1+r\cos\theta \\ y=r\sin\theta \end{matrix}\right] = \int_0^1 dr\, r\int_0^{2\pi}(1+2r\cos\theta+r^2)\,d\theta =$

$=\int_0^1 dr\, r\cdot\int_0^{2\pi}(1+r^2)\,d\theta = 2\pi\int_0^1 (r^3+r^2)\,dr = 2\pi\left[\frac14 r^4+\frac12 r^2\right]_0^1 =$

$= 2\pi\left(\frac12+\frac14\right)=\frac{3\pi}{2}.$

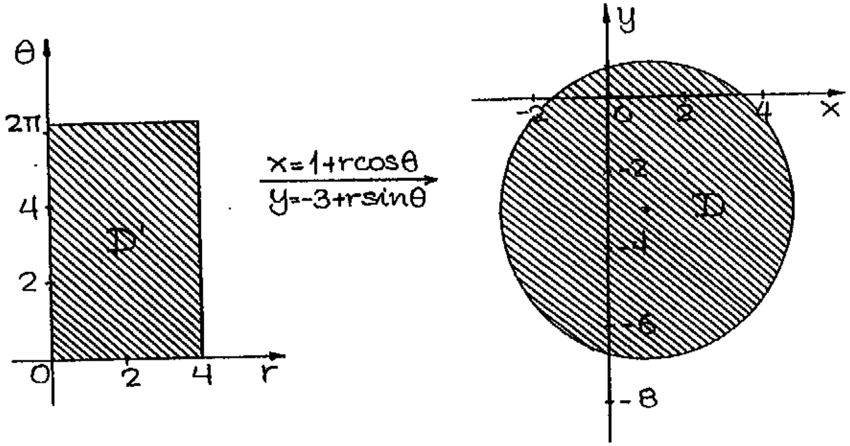
Anm. $\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0.$

Övning 6.25 (s. 115)

$x^2+y^2-2x+6y \equiv (x^2-2x+1)+(y^2+6y+9)-1-9 \equiv (x-1)^2+$

$+(y+3)^2-10 \Rightarrow \underline{D=\{(x,y):\ (x-1)^2+(y+3)^2\le 16\}}.$

D är en disk med centrum i punkten (1,-3) och radien 4 (se fig. på nästa sida).



$\iint_D xy\,dxdy \left[\begin{matrix} x=1+r\cos\theta \\ y=-3+r\sin\theta \end{matrix}\right] = \iint_{D'} (1+r\cos\theta)(-3+r\sin\theta)\,r\,drd\theta =$

$=\iint_{D'} (r^2\sin\theta\cos\theta-3r\cos\theta+r\sin\theta-3)\,r\,drd\theta =$

$=\int_0^4 dr\,r\int_0^{2\pi} d\theta\,(r^2\sin\theta\cos\theta-3r\cos\theta+r\sin\theta-3) =$

$=\int_0^4 dr\,r\int_0^{2\pi}(-3)\,d\theta = -6\pi\int_0^4 r\,dr = -3\pi\left[r^2\right]_0^4 = \underline{-48\pi.}$

Övning 6.26 (s. 116)

$2x^2+3y^2\le 1 \Leftrightarrow \frac{x^2}{(1/\sqrt2)^2}+\frac{y^2}{(1/\sqrt3)^2}\le 1 \Rightarrow \begin{cases} x=\frac{1}{\sqrt2}r\cos\theta \\ y=\frac{1}{\sqrt3}r\sin\theta \end{cases}$

$x^2+y^2=\frac12 r^2\cos^2\theta+\frac13 r^2\sin^2\theta\ ;$

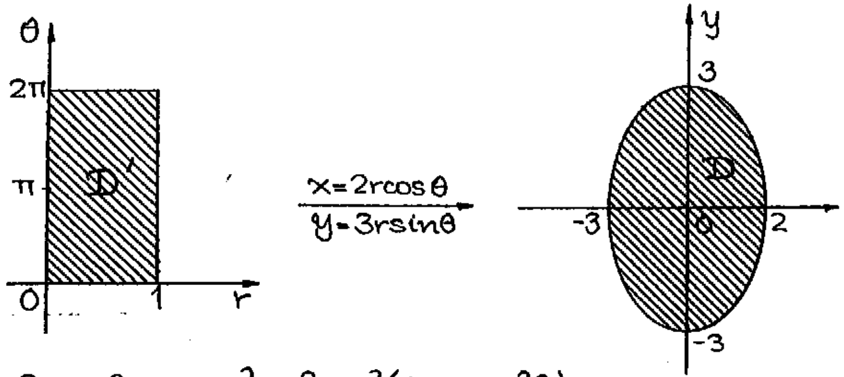
$D=\{(x,y):\ 2x^2+3y^2\le 1\} \Rightarrow D'=[0,1]\times[0,2\pi].$

$\iint_D (x^2+y^2)\,dxdy = \iint_{D'} \frac16 r^2(2+\cos^2\theta)\frac{1}{\sqrt6}r\,drd\theta =$

$=\frac{1}{6\sqrt6}\int_0^1 r^3dr\int_0^{2\pi}\left(\frac52+\frac12\cos2\theta\right)d\theta = \frac{1}{24\sqrt6}\cdot\frac52\cdot 2\pi = \frac{5\sqrt6\pi}{144}.$

Anm. $\int_0^{2\pi}\cos2\theta\,d\theta = 0$

Övning 6.27 (s.116)

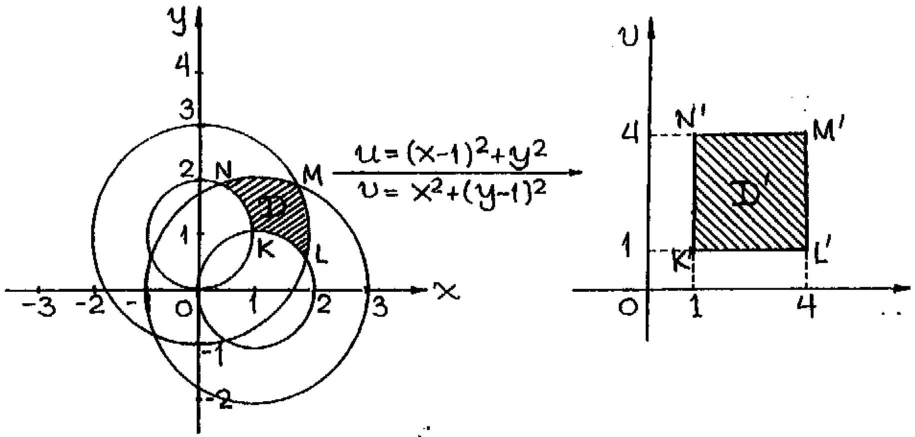


$$\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases} \Rightarrow \begin{cases} x^2+y^2 = r^2(9-5\cos^2\theta) \\ \dfrac{d(x,y)}{d(r,\theta)} = 6r \end{cases}, \ D' = [0,1]\times[0,2\pi].$$

$$\iint_D (x^2+y^2)\,dx\,dy \begin{bmatrix} x=2r\cos\theta \\ y=3r\sin\theta \end{bmatrix} = \iint_{D'} r^2(9-5\cos^2\theta)6r\,dr\,d\theta =$$

$$= \int_0^1 6r^3\,dr \int_0^{2\pi} \left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta = \frac{3}{2}\cdot\frac{13}{2}\cdot 2\pi = \frac{39\pi}{2}.$$

Övning 6.28 (s.116)



$$D = \{ \mathbf{x} : 1 \le (x-1)^2+y^2 \le 4 \} \cap \{(x,y) : 1 \le x^2+(y-1)^2 \le 4\};$$

$$D' = \{(u,v) : 1 \le u \le 4, 1 \le v \le 4\} = [1,4]^2.$$

$$\begin{cases} u = (x-1)^2+y^2 \\ v = x^2+(y-1)^2 \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 2(x-1) & 2y \\ 2x & 2(y-1) \end{vmatrix} = 4(1-x-y) \Rightarrow$$

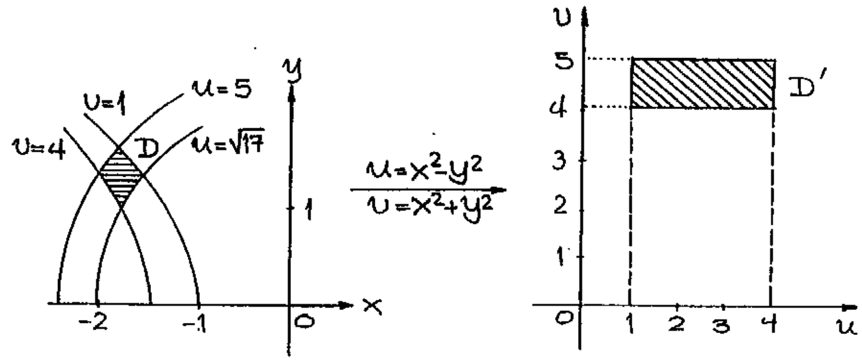
$$\Rightarrow \forall \mathbf{x} \in D : \frac{d(x,y)}{d(u,v)} = \frac{1}{4(1-x-y)}; \quad \underline{\text{Anm.}}\ \frac{d(x,y)}{d(u,v)} < 0.$$

$$x^2+y^2-2x+3 \equiv (x-1)^2+y^2+2 = u+2;$$

$$\iint_D \frac{1-x-y}{x^2+y^2-2x+3}\,dx\,dy = \iint_{D'} \frac{\cancel{1-x-y}}{u+2}\,\frac{-1}{4(\cancel{1-x-y})}\,du\,dv =$$

$$= \frac{1}{4}\iint_{D'} \frac{-1}{u+2}\,du\,dv = -\frac{1}{4}\int_1^4 \frac{1}{u+2}\,du \int_1^4 dv = -\frac{3}{4}\ln 2.$$

Övning 6.28 (s.116)



$$\begin{cases} u = x^2-y^2 \\ v = x^2+y^2 \end{cases} \Leftrightarrow \begin{cases} 2x^2 = u+v \\ 2y^2 = v-u \end{cases} \Leftrightarrow \begin{cases} x = -\sqrt{(u+v)/2} \\ y = \sqrt{(-u+v)/2} \end{cases}; \quad (*)$$

$$\frac{d(u,v)}{d(x,y)} = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 8xy \overset{(*)}{=} -4\sqrt{v^2-u^2} \Rightarrow \frac{d(x,y)}{d(u,v)} = \frac{1}{4\sqrt{v^2-u^2}}$$

forts.

$D=\{(x,y): 1\le x^2-y^2\le 4,\ \sqrt{17}\le x^2+y^2< 5\}$;

$D'=\{(u,v): 1\le u\le 4,\ \sqrt{17}\le v\le 5\}=[1,4]\times[\sqrt{17},5]$.

$$\iint_D (x^4-y^4)\,dxdy \begin{bmatrix} u=x^2-y^2 \\ v=x^2+y^2 \end{bmatrix} = \iint_{D'} uv\cdot\frac{1}{4\sqrt{v^2-u^2}}\,dudv =$$

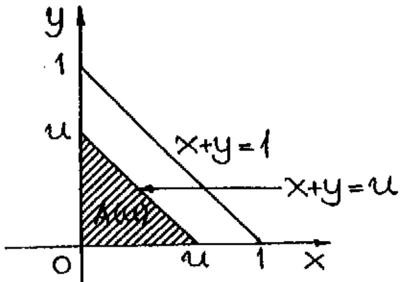
$$=\frac{1}{4}\int_1^4 du\, u\int_{\sqrt{17}}^5 \frac{v}{\sqrt{v^2-u^2}}\,dv = \frac{1}{4}\int_1^4 du\, u\left[\sqrt{v^2-u^2}\right]_{\sqrt{17}}^5 =$$

$$=\frac{1}{4}\int_1^4 u\left(\sqrt{25-u^2}-\sqrt{17-u^2}\right)du = \frac{1}{12}\left[(17-u^2)^{3/2}-(25-u^2)^{3/2}\right]_1^4 =$$

$$=\frac{1}{12}(1-9\cdot 3-16\cdot 4+24\sqrt{24})=\frac{1}{12}(48\sqrt{6}-90)=4\sqrt{6}-\frac{15}{2}.$$

Integration med hjälp av nivåkurvor

Övning 6.30 (s. 116)



$A(u)=\frac{1}{2}u\cdot u=\frac{u^2}{2}\Rightarrow A'(u)=u$; (*)

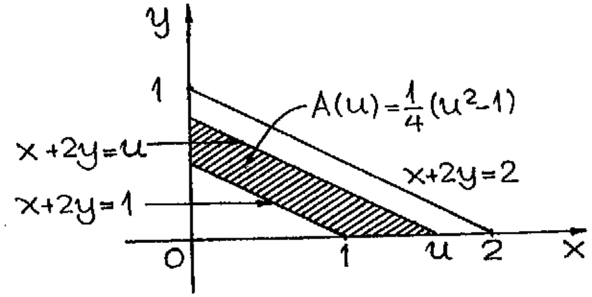
$D=\{(x,y): x+y\le 1,\ x\ge 0,\ y\ge 0\}$.

$$\iint_D e^{-(x+y)^2}\,dxdy \overset{(*)}{=} \int_0^1 e^{-u^2}u\,du = -\frac{1}{2}\left[e^{-u^2}\right]=\frac{1-e^{-1}}{2}.$$

Anm. $dA=A(u+du)-A(u)=\frac{1}{2}(u+du)^2-\frac{1}{2}u^2=$

$$=\frac{1}{2}(2u\,du+(du)^2)\approx u\,du \Leftrightarrow \frac{dA}{du}=u.$$

Övning 6.31 (s. 116)

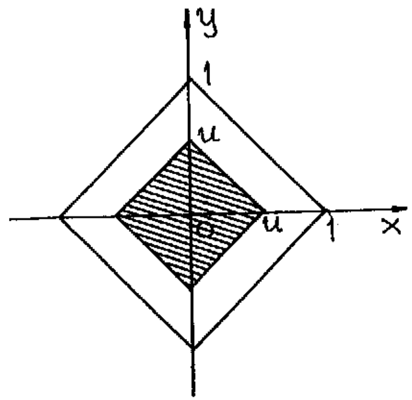


$D=\{(x,y): 1\le x+2y\le 2,\ x\ge 0, y\ge 0\}$; $A'(u)=\frac{1}{2}u$;

$$\iint_D \frac{1}{(1+(x+2y)^2)^2}\,dxdy=\int_1^2 \frac{1}{(1+u^2)^2}\cdot\frac{u}{2}\,du=\frac{1}{4}\int_1^2\frac{d(u^2+1)}{(1+u^2)^2}=$$

$$=-\frac{1}{4}\left[\frac{1}{u^2+1}\right]_1^2=\frac{1}{4}\left(\frac{1}{2}-\frac{1}{5}\right)=\frac{3}{40}.$$

Övning 6.32 (s. 116)



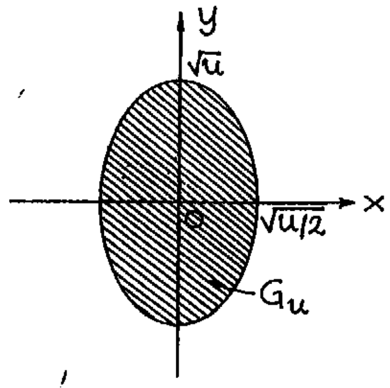
$A(u)=(\sqrt{2}u)^2=2u^2\Rightarrow A'(u)=4u$;

$$\iint_D (|x|+|y|)^2\,dxdy=\int_0^1 u^2\cdot 4u\,du=\int_0^1 4u^3\,du=1.$$

Det skuggade området är en kvadrat med sidan $\sqrt{2}u$.

Generaliserade dubbelintegraler

Övning 6.33 (s.117)



$G_u = \{(x,y): 2x^2+y^2 \le u\} \Rightarrow \bigcup_u G_u = \mathbb{R}^2.$

$\mu(G_u) = A_u = \pi\cdot\sqrt{\frac{u}{2}}\cdot\sqrt{u} = \frac{\pi}{\sqrt2}u \Rightarrow A'_u = \frac{\pi}{\sqrt2};$

$$\iint_{G_u} \frac{(2x^2+y^2)}{1+(2x^2+y^2)^4}\,dxdy = \int_0^u \frac{t}{1+t^4}\cdot\frac{\pi}{\sqrt2}\,dt = \frac{\pi}{2\sqrt2}\int_0^u \frac{(t^2)'}{1+(t^2)^2}\,dt$$

$$= \frac{\pi}{2\sqrt2}\int_0^{u^2}\frac{1}{1+\tau^2}\,d\tau = \frac{\pi}{2\sqrt2}[\arctan\tau]_0^{u^2} = \frac{\pi}{2\sqrt2}\arctan u^2 \Rightarrow$$

$$\Rightarrow \iint_{\mathbb{R}^2}\frac{2x^2+y^2}{1+(2x^2+y^2)^4}\,dxdy = \lim_{u\to\infty}\frac{\pi}{2\sqrt2}\arctan u^2 = \frac{\pi^2}{4\sqrt2}.$$

Anm. Författarna föreslår polär substitution. Se vad de har att komma med på s. 130.

Övning 6.34 (s.117)

$C_R = \{(x,y): x^2+y^2 \le R^2\} \Rightarrow \mathbb{R}^2 = \bigcup_R C_R.$

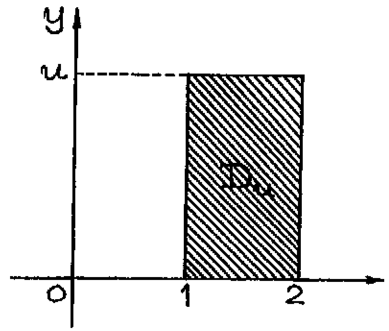
$$\iint_{C_R} x^2 e^{-\sqrt{x^2+y^2}}\,dxdy \begin{bmatrix}x=r\cos\theta\\ y=r\sin\theta\end{bmatrix} = \int_0^R r^3e^{-r}\,dr\int_0^{2\pi}\cos^2\theta\, d\theta =$$

$$= \pi\int_0^R r^3e^{-r}\,dr = -\pi[e^{-r}(r^3+3r^2+6r+6)]_0^R = 6\pi -$$

$$-\pi e^{-R}(R^3+3R^2+6R+6) \Rightarrow \iint_{\mathbb{R}^2} x^2e^{-(x^2+y^2)}\,dxdy =$$

$$= 6\pi - \lim_{R\to\infty}\pi e^{-R}(R^3+3R^2+6R+6) = 6\pi.$$

Övning 6.35 (s.117)



$D_u = \{(x,y): 1\le x\le 2,\ 0\le y\le u\} \Rightarrow D = \bigcup_u D_u.$

$$\iint_{D_u}\frac{1}{1+x^2y^2}\,dxdy = \int_1^2 dx\int_0^u\frac{1}{1+x^2y^2}\,dy = \int_1^2 B(x)\,dx;$$

$$B(x) = \int_0^u \frac{1}{1+(xy)^2}\,dy\ [t=xy] = \int_0^{xu}\frac1x\,\frac{1}{t^2+1}\,dt =$$

$$= \frac1x[\arctan t]_0^{xu} = \frac1x\arctan(xu);$$

$$\iint_D \frac{1}{1+x^2y^2}\,dxdy = \lim_{u\to\infty}\int_1^2\frac1x\arctan(xu)\,dx = (*) =$$

$$= \int_1^2\frac1x\left(\lim_{u\to\infty}\arctan(xu)\right)dx = \frac\pi2\int_1^2\frac1x\,dx = \frac\pi2[\ln x]_1^2 =$$

$$= \frac\pi2(\ln2-\ln1) = \frac\pi2\ln2.$$

Anm. $|B(x)| = \frac1x|\arctan(xu)| \le \frac\pi2\cdot\frac1x$ så gränsövergången har mening i (*) ovan.

Övning 6.36 (s. 117)

$A_u = \{(x,y): x^2+y^2 \le u^2\} \Rightarrow \mathbb{R}^2 = \bigcup_u A_u.$

$I(\alpha) = \iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^\alpha}\,dxdy = \lim_{u\to\infty} \iint_{A_u} \frac{dxdy}{(1+x^2+y^2)^\alpha} =$

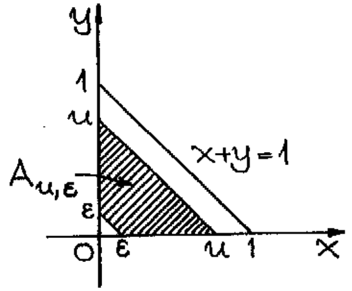
$= \lim_{u\to\infty} \int_0^u \frac{r}{(1+r^2)^\alpha}dr \int_0^{2\pi} d\theta = \pi \int_0^\infty \frac{r}{(1+r^2)^\alpha}dr \ [t=r^2+1] =$

$= \frac{\pi}{2}\int_1^\infty \frac{1}{t^\alpha}dt = \frac{\pi}{2}\lim_{u\to\infty}\left[\frac{-1}{\alpha-1}\frac{1}{t^{\alpha-1}}\right]_1^u = \frac{\pi}{2}\lim_{u\to\infty}\frac{1}{\alpha-1}(1-$

$- u^{1-\alpha}) = \frac{\pi}{\alpha-1}$ för $\alpha-1>0 \Leftrightarrow \alpha>1.$

Resultat: Integralen konvergerar för $\alpha>1$; dess värde är $\frac{\pi}{\alpha-1}$.

Övning 6.37 (s. 117)



$A_{u,\varepsilon} = \{(x,y): \varepsilon \le x+y \le u < 1\} \Rightarrow D = \bigcup_\varepsilon \bigcup_u A_{u,\varepsilon}.$

$\mu(A_{u,\varepsilon}) = \frac{1}{2}(u^2-\varepsilon^2) \Rightarrow \frac{d}{du}\mu(A_{u,\varepsilon}) = u.$

$\iint_D \frac{1}{x+y}dxdy = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{u}u\,du = \lim_{\varepsilon\to 0}\int_\varepsilon^1 du =$

$= \lim_{\varepsilon\to 0}(1-\varepsilon) = 1.$ (Gå nu till facit, s. 131).

Övning 6.38 (s. 117)

Se figuren i föregående uppgift.

$I(\alpha) = \iint_D \frac{1}{(x+y)^\alpha}dxdy = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{1}{u^\alpha}u\,du = \lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{du}{u^{\alpha-1}}$

$= \lim_{\varepsilon\to 0}\left[\frac{1}{\alpha-2}\cdot\frac{1}{u^{\alpha-2}}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0}\frac{1}{\alpha-2}(1-\varepsilon^{2-\alpha}) \overset{\alpha<2}{=} \frac{1}{\alpha-2}.$

Resultat: Integralen konvergerar för $\alpha<2$.

Övning 6.39 (s. 117)

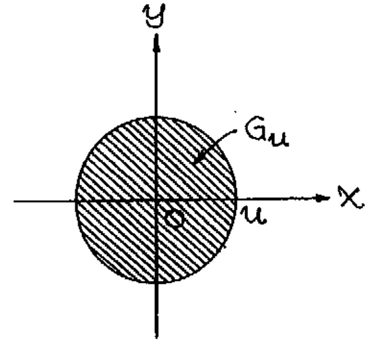
Se figuren till 6.30!

$I(\alpha) = \iint_D (1-x-y)^y dxdy = \int_0^1 (1-u)^\alpha u\,du =$

$= \left[-\frac{(1-u)^{\alpha+1}}{\alpha+1}u\right]_0^1 + \int_0^1 \frac{(1-u)^{\alpha+1}}{\alpha+1}du = \frac{1}{\alpha+1}\int_0^1 (1-u)^{\alpha+1}du$

$= \frac{-1}{(1+\alpha)(2+\alpha)}\left[(1-u)^{\alpha+2}\right]_0^1 = \frac{1}{(\alpha+1)(\alpha+2)},\ \alpha>-1.$

Övning 6.40 (s. 117)



$G_u = \{(x,y): \sqrt{x^2+y^2} < u\} \Rightarrow \bigcup_u G_u = \mathbb{R}^2.$

forts.

$A(u) = \mu(G_u) = \pi u^2 \Rightarrow A'(u) = 2\pi u.$

$$\iint_D \frac{e^{-\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}dxdy = \int_0^\infty \frac{e^{-u}}{u}\cdot 2\pi u\,du = 2\pi\int_0^\infty e^{-u}du =$$
$$= \lim_{R\to\infty}[e^{-u}]_0^R \cdot 2\pi = \underline{2\pi}.$$

Anm. Jag har använt metoden med nivåkurvor; författarna väljer polär substitution.

Övning 6.41 (s. 118)

Med samma figur som i föregående övning och med samma metod får vi

$$\iint_{\mathbb{R}^2}\frac{1}{(x^2+y^2)^{\alpha/2}}dxdy = \lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_\varepsilon^R \frac{1}{u^\alpha}2\pi u\,du =$$
$$= 2\pi\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_\varepsilon^R u^{1-\alpha}du = 2\pi(\lim_{\varepsilon\to 0}\lim_{R\to\infty}\left[\frac{u^{2-\alpha}}{2-\alpha}\right]_\varepsilon^R =$$
$$= 2\pi\left(\lim_{R\to\infty}\frac{R^{2-\alpha}}{2-\alpha} - \lim_{\varepsilon\to 0}\frac{\varepsilon^{2-\alpha}}{2-\alpha}\right) < \infty \Rightarrow 2-\alpha>0 \wedge 2-\alpha<0.$$

Det finns inga α s.a. $\alpha<2 \wedge \alpha>2$.

Resultat: Integralen är divergent för alla α.

Övning 6.42 (s. 118)

$\forall x\in\mathbb{R}^2$: $|\arctan(xy)| \le \frac{\pi}{2}$.

$$\left|\iint_D \frac{\arctan^2(xy)}{(x^2+y^2)^2}dxdy\right| \le \iint_D \frac{|\arctan(xy)|^2}{(x^2+y^2)^2}dxdy \le$$
$$\le \iint_D \frac{\pi^2/4}{(x^2+y^2)^2}dxdy < \infty \quad (\alpha=4 \text{ i föreg. övning}, \varepsilon=1).$$

Övning 6.43 (s. 118)

$$f(x,y) = \frac{x^2}{(x^2+1)(x^2+y^2)^{3/2}};$$

$$\iint_{\mathbb{R}^2} f(x,y)\,dxdy = \iint_{|x|<1} f(x,y)\,dxdy + \iint_{|x|\ge 1} f(x,y)\,dxdy;$$

(i) $$\iint_{|x|<1}\frac{x^2}{(x^2+1)(x^2+y^2)^{3/2}}dxdy \le \iint_{|x|<1}\frac{x^2+y^2}{(x^2+1)(x^2+y^2)^{3/2}}dxdy =$$
$$\le \iint_{|x|<1}\frac{1}{(x^2+1)(x^2+y^2)^{1/2}}dxdy \le \iint_{|x|\ge 1}\frac{1}{(x^2+y^2)^{1/2}}dxdy < \infty, (\alpha<2).$$

(ii) $$\iint_{|x|\ge 1}\frac{x^2}{(x^2+1)(x^2+y^2)^{3/2}}dxdy \le \iint_{|x|\ge 1}\frac{1}{(x^2+y^2)^{3/2}}dxdy =$$
$$= \int_1^\infty \frac{1}{r^3}\cdot 2\pi r\,dr = 2\pi\int_1^\infty \frac{1}{r^2}dr < \infty.$$

Resultat: Integralen är konvergent.

Övning 6.44 (s. 118)

$\forall x\in\mathbb{R}\setminus\{0\}$: $\frac{x^2+1}{x^2} > 1$.

$D_\varepsilon = \{(x,y):\ \varepsilon^2 \le x^2+y^2 \le 1\}$, $D = \{(x,y):\ x^2+y^2\le 1\}$.

$$\iint_{D_\varepsilon}\frac{x^2+1}{x^2(x^2+y^2)^{3/2}}dxdy \ge \iint_{D_\varepsilon}\frac{1}{(x^2+y^2)^{3/2}}dxdy =$$
$$= \int_\varepsilon^1 \frac{1}{r^3}2\pi r\,dr = \int_\varepsilon^1 \frac{2\pi}{r^2}dr = -2\pi\left[\frac{1}{r}\right]_\varepsilon^1 = 2\pi\left(\frac{1}{\varepsilon}-1\right)\xrightarrow[\varepsilon\to 0]{}\infty.$$

Resultat: Integralen är divergent.

Anm. Man kan inte göra jämförelse med D. Origo (där) kallas här egentlig singularitet.

Blandade problem

Övning 6.45 (S. 118)

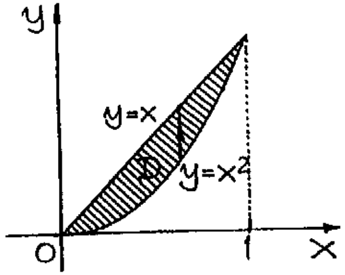
$$\begin{cases} u = x^3 \\ v = y^3 \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)} = 9x^2y^2 \;;\; D = \{(x,y) : x^2+y^2<1\}.$$

$$\mu(A) = \iint_A du\,dv = \iint_D \left|\frac{d(u,v)}{d(x,y)}\right| dx\,dy = \iint_{|x|\le 1} 9x^2y^2\,dx\,dy =$$

$$= \int_0^1 9r^5\,dr \int_0^{2\pi} \cos^2\theta \sin^2\theta\,d\theta = \frac{3}{2}\cdot\int_0^{2\pi}\cdot\frac{1}{4}\sin^2 2\theta\,d\theta =$$

$$= \frac{3}{2}\int_0^{2\pi}\frac{1}{8}(1-\cos 4\theta)\,d\theta = \frac{3}{2}\cdot\frac{1}{8}\cdot 2\pi = \frac{3\pi}{8} \approx \underline{1{,}178\text{ ae.}}$$

Övning 6.46 (S. 118)



$$\iint_D x e^{-2y}\,dx\,dy = \int_0^1 dx\, x\int_{x^2}^{x} e^{-2y}\,dy = \int_0^1\left(\left[-\frac{x}{2}e^{-2y}\right]_{x^2}^{x}\right)dx$$

$$= \int_0^1 \frac{x}{2}(e^{-2x^2}-e^{-2x})\,dx = \frac{1}{2}\int_0^1 x e^{-2x^2}dx - \frac{1}{2}\int_0^1 x e^{-2x}dx =$$

$$= -\frac{1}{8}\left[e^{-2x^2}\right]_0^1 - \frac{1}{2}\int_0^1 x e^{-2x}dx = \frac{1-e^{-2}}{8} + \frac{1}{4}\left[x e^{-2x}\right]_0^1 -$$

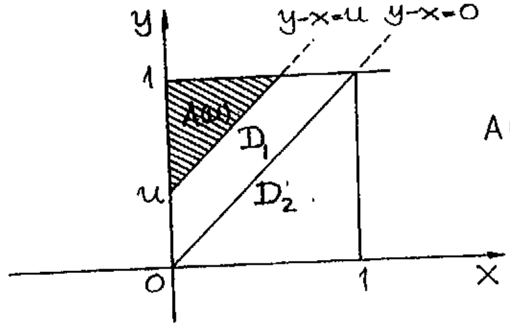
$$-\frac{1}{4}\int_0^1 e^{-2x}dx = \frac{1-e^{-2}}{8} + \frac{1}{4}e^{-2} + \underbrace{\frac{1}{8}\left[e^{-2x}\right]_0^1}_{\frac{e^{-2}-1}{8}} = \frac{1}{4}e^{-2}.$$

<u>Resultat</u>: Se ovan!

Övning 6.47 (S. 118)

$f(x,y) = |x-y|^{-1/2} = |y-x|^{-1/2} = f(y,x)$;

f är spegelsymmetrisk m.a.p. linjen $y=x$; denna linje är singulär för f. Jag kommer att utnyttja symmetrin för f för att beräkna integralen med hjälp av metoden med <u>nivåkurvor</u>.



$A(u) = \frac{(1-u)^2}{2}$.

$D_1 = \{(x,y)\in[0,1]^2 : y \ge x\}$, $D_2 = \{(x,y)\in[0,1] : y < x\}$.

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(x,y) + \iint_{D_2} f(x,y)\,dx\,dy =$$

$$= 2\iint_{D_1} f(x,y)\,dx = 2\int_1^{\varepsilon}\frac{1}{\sqrt{u}}\cdot(u-1)\,du = 2\int_1^{\varepsilon}(u^{1/2}-u^{-1/2})\,du$$

$$= 2\left[\frac{2}{3}u\sqrt{u} - 2\sqrt{u}\right]_1^{\varepsilon} = \frac{4}{3}\varepsilon\sqrt{\varepsilon} - 4\sqrt{\varepsilon} - \frac{4}{3} + 4 \xrightarrow[\varepsilon\to 0]{} 4 - \frac{4}{3} = \frac{8}{3}.$$

Övning 6.48 (S. 119)

$f(x,y) = xye^{-xy} \Rightarrow f(x,\frac{k}{x}) = ke^{-k}$; denna går ej mot 0 då $|x|\to\infty$. Integralen är divergent.

Övning 6.49 (s.119)

a) $\displaystyle\iint_{|x|\le\sqrt{2}} x^2 e^{(x^2+y^2)^2}dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \int_0^{\sqrt{2}} r^3 e^{r^4}dr \int_0^{2\pi}\cos^2\theta\, d\theta =$

$\displaystyle= \frac{1}{8}\int_0^{\sqrt{2}} 4r^3 e^{r^4}dr \int_0^{2\pi}(1+\cos 2\theta)d\theta = \frac{1}{4}\left[e^{r^4}\right]_0^{\sqrt{2}}\cdot\pi = \frac{\pi}{4}(e^4-1).$

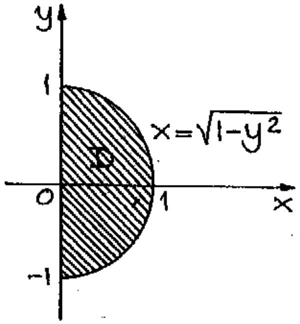
b) $\displaystyle\iint_{|x|\le\sqrt{2}} y^2 e^{(x^2+y^2)}dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \int_0^{\sqrt{2}} r^3 e^{r^4}dr \int_0^{2\pi}\sin^2\theta\, d\theta =$

$\displaystyle= \frac{1}{8}\int_0^{\sqrt{2}} 4r^3 e^{r^4}dr \int_0^{2\pi}(1-\cos 2\theta)d\theta = \frac{1}{4}\left[e^{r^4}\right]_0^{\sqrt{2}}\cdot\pi = \frac{\pi}{4}(e^4-1).$

c) $\displaystyle\iint_{|x|\le\sqrt{2}} (x^2+y^2)e^{-(x^2+y^2)}dxdy = \iint_{|x|\le 2} x^2 e^{-(x^2+y^2)^2}dxdy +$

$\displaystyle+ \iint_{|x|\le\sqrt{2}} y^2 e^{-(x^2+y^2)^2}dxdy = 2\cdot\frac{\pi}{4}(e^4-1) = \frac{\pi}{2}(e^4-1).$

Övning 6.50 (s.119)



$\displaystyle\iint_D \sqrt{1-y^2}\,dxdy \begin{bmatrix} x=r\cos\theta \\ y=r\sin\theta \end{bmatrix} = \int_{-\pi/2}^{\pi/2}\Big(\underbrace{\int_0^1 \sqrt{1-r^2\sin^2\theta}\; r\,dr}_{A(\theta)}\Big)d\theta$

$\displaystyle A(\theta) = \int_0^1 \sqrt{1-r^2\sin^2\theta}\; r\,dr \begin{bmatrix} z^2 = 1-r^2\sin^2\theta \\ z\,dz = -r\,dr\sin^2\theta \end{bmatrix} =$

$\displaystyle= \int_1^{\cos\theta} z\cdot\left(-\frac{z\,dz}{\sin^2\theta}\right) = \frac{-1}{\sin^2\theta}\int_1^{\cos\theta} z^2 dz = \frac{1}{3}\,\frac{1-\cos^3\theta}{\sin^2\theta} =$

$\displaystyle= \frac{1}{3}\,\frac{1-\cos\theta(1-\sin^2\theta)}{\sin^2\theta} = \frac{1}{3}\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta} + \cos\theta\right) \Rightarrow$

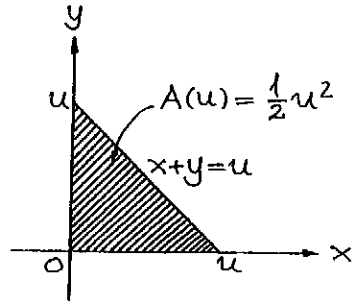
$\displaystyle\Rightarrow \iint_D \sqrt{1-y^2}\,dxdy = \frac{1}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta} + \cos\theta\right)d\theta =$

$\displaystyle= \frac{1}{3}\left[-\cot\theta + \frac{1}{\sin\theta} + \sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{1}{3}(0+1+1+0+1+1) = \frac{4}{3}.$

Anm. $\displaystyle\iint_D \sqrt{1-y^2}\,dxdy = \int_{-1}^{1}\Big(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx\Big)dy =$

$\displaystyle= \int_{-1}^{1}(1-y^2)dy = 2\int_0^1 (1-y^2)dy = 2\left[y - \frac{1}{3}y^3\right]_0^1 = \frac{4}{3}.$

Övning 6.51 (s.119)



$A(u) = \frac{1}{2}u^2 \Rightarrow A'(u) = 2u$

$\displaystyle\iint_{[0,\infty[^2} \frac{1}{1+(x+y)^4}dxdy = \int_0^\infty \frac{u}{1+u^4}du = \int_0^\infty \frac{u}{1+(u^2)^2}du \;[v=u^2] =$

$\displaystyle= \frac{1}{2}\int_0^\infty \frac{1}{1+v^2}dv = \frac{1}{2}\lim_{R\to\infty}\left[\arctan v\right]_0^R = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$

Övning 6.52 (s.119)

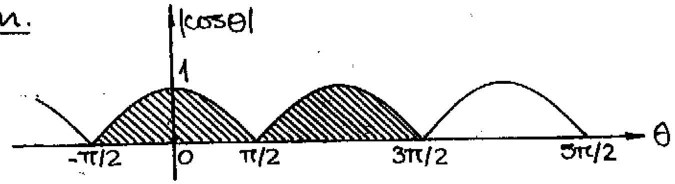
$\begin{cases} u = \frac{3}{5}x + \frac{4}{5}y \\ v = -\frac{4}{5}x + \frac{3}{5}y \end{cases} \Rightarrow \begin{cases} u^2+v^2 = x^2+y^2 \\ \dfrac{d(u,v)}{d(x,y)} = 1 \end{cases}$ (ren rotation).

$D=\{(x,y): x^2+y^2\le 2\} \Rightarrow D'=\{(u,v): u^2+v^2\le 2\}$.

$$\iint_D |3x+4y|\,dxdy \begin{bmatrix}u=\frac35x+\frac45y\\ v=-\frac45x+\frac35y\end{bmatrix} = \iint_{D'} 5|u|\,dudv =$$

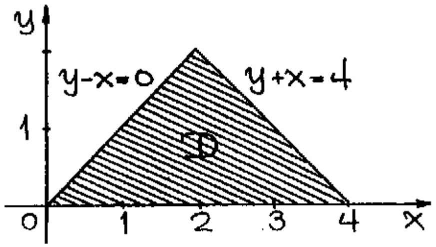
$$= \begin{bmatrix}u=r\cos\theta\\ v=r\sin\theta\end{bmatrix} = \int_0^{\sqrt2} 5r^2dr \int_{-\pi/2}^{3\pi/2}|\cos\theta|\,d\theta = \frac53\cdot 2\sqrt2\cdot 4 = \frac{40\sqrt2}{3}.$$

<u>Anm.</u>



Varje "kulle" har arean 2ae.

Övning 6.53 (s. 119)

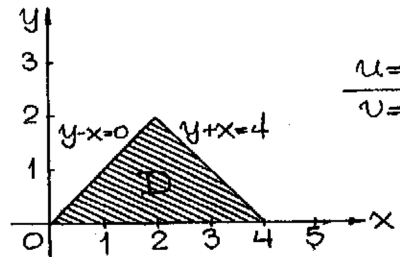


$$\begin{cases}u=y-x\\ v=y+x\end{cases} \Leftrightarrow \begin{cases}x=\frac12(v-u)\\ y=\frac12(v+u)\end{cases} \Rightarrow \frac{d(x,y)}{d(u,v)}=\begin{vmatrix}-1/2 & 1/2\\ 1/2 & 1/2\end{vmatrix}=-\frac12.$$

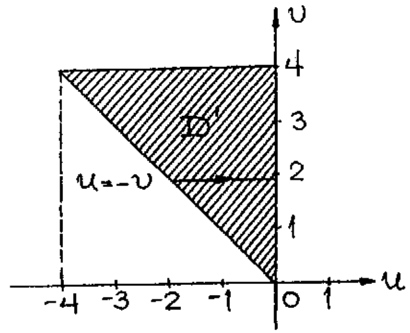
$D=\{(x,y): y-x\le 0 \wedge y+x\le 4 \wedge y\ge 0\}$;

$D'=\{(u,v): u\le 0 \wedge v\le 4 \wedge v+u\ge 0\} =$

$=\{(u,v): -v\le u\le 0,\ v\le 4\}$. (Se nästa sida)



$$\xrightarrow[v=y+x]{u=y-x}$$



$$\iint_D e^{(y-x)/(y+x)}\,dxdy \begin{bmatrix}u=y-x\\ v=y+x\end{bmatrix} = \iint_{D'} e^{u/v}(-\tfrac12)\,dudv =$$

$$=-\frac12\int_0^4\left(\int_{-v}^0 e^{u/v}du\right)dv = -\frac12\int_0^4\left(\left[ve^{u/v}\right]_{u=-v}^{u=0}\right)dv =$$

$$=\frac12\int_0^4(1-e^{-1})v\,dv = \frac14\cdot 4^2(1-e^{-1}) = \underline{4(1-e^{-1})}.$$

Övning 6.54 (s. 120)

$D=\{(x,y): x\ge 0,\ 1\le y\le 2\}$.

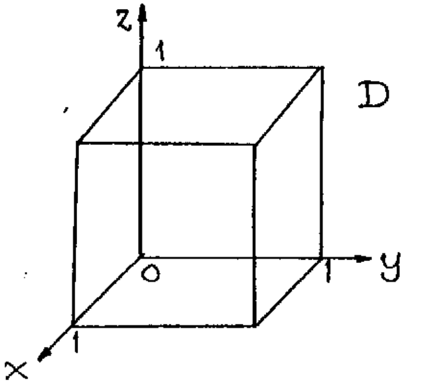
$$\iint_D e^{-xy}\,dxdy = \int_1^2\left(\int_0^\infty e^{-xy}dx\right)dy = \int_1^2\left(\left[-\frac{e^{-xy}}{y}\right]_0^\infty\right)dy =$$

$$=\int_1^2 \frac1y\,dy = [\ln y]_1^2 = \ln 2 \Rightarrow \int_0^\infty \frac{e^{-x}-e^{-2x}}{x}\,dx =$$

$$=\int_0^\infty\left(\int_1^2 e^{-xy}dy\right)dx = \iint_D e^{-xy}\,dxdy = \ln 2.$$

7. Trippelintegraler

Övning 7.1 (s. 132)



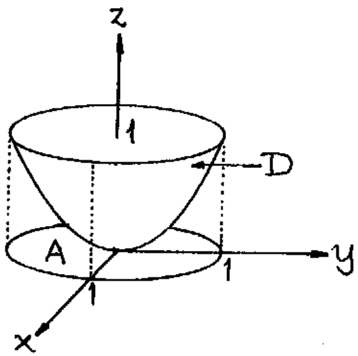
$D = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\} = [0,1]^3.$

D går under namnet "enhetskuben".

$$\iiint_D e^{x+y+z}\,dx\,dy\,dz = \int_0^1 e^x dx \int_0^1 e^y dy \int_0^1 e^z dz = (e-1)^3.$$

Övning 7.2 (s. 132)

$D = \{(x,y,z) : x^2+y^2 \le z \le 1\}$, $A = \{(x,y) : x^2+y^2 \le 1\}$.



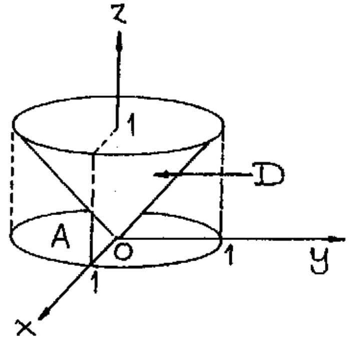
$$\iiint_D z\sqrt{x^2+y^2}\,dx\,dy\,dz = \iint_A \sqrt{x^2+y^2}\Big(\int_{x^2+y^2}^{1} z\,dz\Big)dx\,dy =$$

$$= \frac{1}{2}\iint_A \sqrt{x^2+y^2}\big([z^2]_{x^2+y^2}^{1}\big)dx\,dy = \frac{1}{2}\iint_A \sqrt{x^2+y^2}\,(1-(x^2+y^2)^2)\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 r(1-r^4)\,r\,dr \int_0^{2\pi} d\theta = \pi\int_0^1 (r^2-r^6)\,dr = \pi\Big[\frac{r^3}{3}-\frac{r^7}{7}\Big]_0^1 = \frac{4\pi}{21}.$$

Övning 7.3 (s. 132)

$D = \{(x,y,z) : 0 \le x^2+y^2 \le z^2,\ 0 \le z \le 1\} = \{(x,y,z) : \sqrt{x^2+y^2} \le z \le 1\}$

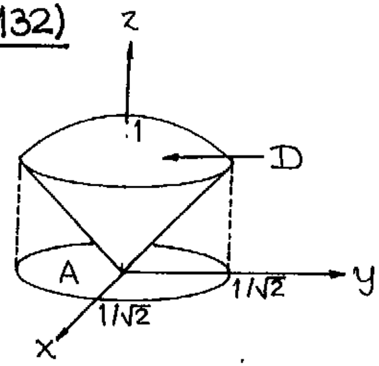


$$\iiint_D (x^2+y^2)\,dx\,dy\,dz = \iint_A (x^2+y^2)\Big(\int_{\sqrt{x^2+y^2}}^{1} dz\Big)dx\,dy =$$

$$= \iint_A (x^2+y^2)(1-\sqrt{x^2+y^2})\,dx\,dy = (\text{polära koordinater}) =$$

$$= \int_0^1 r^2(1-r)\,r\,dr \int_0^{2\pi} d\theta = 2\pi\int_0^1 (r^3-r^4)\,dr = 2\pi\Big[\frac{r^4}{4}-\frac{r^5}{5}\Big]_0^1 = \frac{\pi}{10}.$$

Övning 7.4 (s. 132)



forts.

$D=\{(x,y,z): \sqrt{x^2+y^2}\le z\le\sqrt{1-x^2-y^2}\}$.

$\sqrt{x^2+y^2}=\sqrt{1-x^2-y^2} \Leftrightarrow x^2+y^2=1-x^2-y^2 \Leftrightarrow x^2+y^2=\frac{1}{2}$;

$A=\{(x,y): x^2+y^2\le \frac{1}{2}\}$.

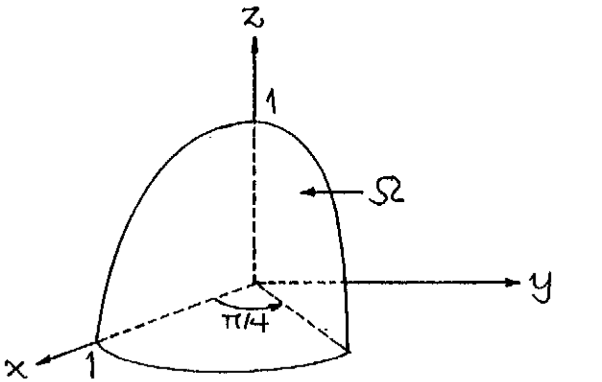
$$\iiint_D \frac{1}{1+x^2+y^2}\,dxdydz = \iint_A \frac{1}{1+x^2+y^2}\left(\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,dz\right)dxdy =$$

$$=\iint_A \frac{1}{1+x^2+y^2}\left(\left[\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}}\right)dxdy = \frac{1}{2}\iint_A \frac{1-2(x^2+y^2)}{1+x^2+y^2}\,dxdy=$$

$$=\frac{1}{2}\int_0^{1/\sqrt{2}}\frac{1-2r^2}{1+r^2}\,r\,dr\int_0^{2\pi}d\theta = \pi\int_0^{1/\sqrt{2}}\left(\frac{3r}{r^2+1}-2r\right)dr=$$

$$=\pi\left[\frac{3}{2}\ln(r^2+1)-r^2\right]_0^{1/\sqrt{2}} = \pi\left(\frac{3}{2}\ln\frac{3}{2}-\frac{1}{2}\right)=\frac{\pi}{2}\left(3\ln\frac{3}{2}-1\right).$$

Övning 7.5 (s. 132)



$\Omega=\{(x,y,z): y\ge 0,\ z\ge 0,\ y\le x,\ z\le\sqrt{4-x^2-y^2}\}$.

Vi inför sfäriska koordinater (r,θ,ϕ) och får

$x^2+y^2=r^2\sin^2\theta$, $J(r,\theta)=r^2\sin\theta$;

$\Omega'=[0,2]\times[0,\frac{\pi}{2}]\times[0,\frac{\pi}{4}]$; forts.

$$\mu(\Omega)=\iiint_\Omega (x^2+y^2)\,dxdydz \left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 2\\ y=r\sin\theta\sin\phi & 0\le\theta\le\frac{\pi}{2} \\ z=r\cos\theta & 0\le\phi\le\frac{\pi}{4}\end{array}\right]=$$

$$=\iiint_{\Omega'} r^2\sin^2\theta\cdot r^2\sin\theta\,drd\theta d\phi = \iiint_{\Omega'} r^4\sin^3\theta\,drd\theta d\phi=$$

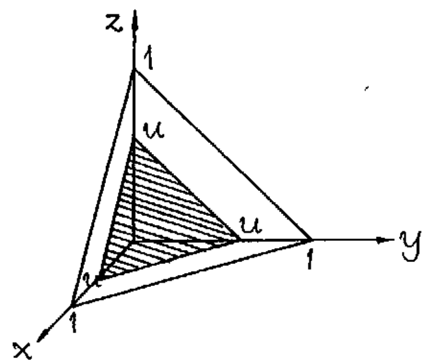
$$=\underbrace{\int_0^2 r^4dr}_{32/5}\int_0^{\pi/2}\sin^3\theta\, d\theta\underbrace{\int_0^{\pi/4}d\phi}_{\pi/4} = \frac{8\pi}{5}\int_0^{\pi/2}\underbrace{(1-\cos^2\theta)\sin\theta\, d\theta}_{u=\cos\theta}=$$

$$=\frac{8\pi}{5}\int_1^0(1-u^2)(-du)=\frac{8\pi}{5}\int_0^1(1-u^2)\,du=\frac{8\pi}{5}\left(1-\frac{1}{3}\right)=\frac{16\pi}{15}.$$

Resultat: Kroppens massa är $\frac{16\pi}{15}\approx 3{,}351$ viktenheter.

Övning 7.6 (s. 132)

Jag kommer att använda metoden med nivåytor.



$G_u=\{(x,y,z): x+y+z\le u<1,\ x\ge 0, y\ge 0, z\ge 0\}$.

$\mu(G_u)=\frac{1}{3}\cdot\frac{1}{2}u^2\cdot u=\frac{1}{6}u^3=V(u)\Rightarrow V'(u)=\frac{1}{2}u^2$.

$$\iiint_D\frac{1}{1+(x+y+z)^3}\,dxdydz=\int_0^1\frac{1}{1+u^3}\cdot\frac{1}{2}u^2\,du=$$

$$=\frac{1}{6}\int_0^1\frac{1}{1+u^3}\cdot(1+u^3)'\,du=\frac{1}{6}\left[\ln(1+u^3)\right]_0^1=\frac{1}{6}\ln 2.$$

Övning 7.7 (s. 132)

Vi inför sfäriska koordinater (r,θ,ϕ) och får

$$D=\{(x,y,z):\ 1\le x^2+y^2+z^2\le 4\}\Rightarrow D'=[1,2]\times[0,\pi]\times[0,2\pi].$$

$$\iiint_D \frac{1}{x^2+y^2+z^2}\,dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 1\le r\le 2\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi\\ z=r\cos\theta & 0\le\phi\le 2\pi\end{array}\right]=$$

$$=\iiint_{D'}\frac{1}{r^2}\cdot r^2\sin\theta\,drd\theta d\phi=\underbrace{\int_1^2 dr}_{1}\underbrace{\int_0^\pi\sin\theta\,d\theta}_{2}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}=4\pi.$$

Övning 7.8 (s. 132)

Vi inför sfäriska koordinater (r,θ,ϕ) och får

$$D=\{(x,y,z):\ x^2+y^2+z^2\le 1\}\Rightarrow D'=[0,1]\times[0,\pi]\times[0,2\pi].$$

a) $$\iiint_D(x^2+y^2+z^2)\,dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi\\ z=r\cos\theta & 0\le\phi\le 2\pi\end{array}\right]=$$

$$=\iiint_{D'}r^2\cdot r^2\sin\theta\,drd\theta d\phi=\underbrace{\int_0^1 r^4dr}_{1/5}\underbrace{\int_0^\pi\sin\theta\,d\theta}_{2}\underbrace{\int_0^{2\pi}d\phi}_{2\pi}=\frac{4\pi}{5}.$$

b) $$\iiint_D x^2\,dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi\\ z=r\cos\theta & 0\le\phi\le 2\pi\end{array}\right]=$$

$$=\iiint_{D'}r^2\sin^2\theta\cos^2\phi\; r^2\sin\theta\,drd\theta d\phi=$$

$$=\int_0^1 r^4dr\int_0^\pi\sin^3\theta\,d\theta\int_0^{2\pi}\cos^2\phi\,d\phi=\frac{1}{5}\cdot\pi\int_0^\pi\sin^3\theta\,d\theta=$$

$$=\frac{\pi}{5}\int_0^\pi(1-\cos^2\theta)\sin\theta\,d\theta=\frac{\pi}{5}\left[\frac{1}{3}\cos^3\theta-\cos\theta\right]_0^\pi=\frac{\pi}{5}\cdot\frac{4}{3}=\frac{4\pi}{15}.$$

c) $$\iiint_D x\,dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi\\ z=r\cos\theta & 0\le\phi\le 2\pi\end{array}\right]=$$

$$=\iiint_{D'}r\sin\theta\cos\phi\; r^2\sin\theta\,drd\theta d\phi=$$

$$=\int_0^1 r^3dr\int_0^\pi\sin^2\theta\,d\theta\underbrace{\int_0^{2\pi}\cos\phi\,d\phi}_{0}=0.$$

Övning 7.9 (s. 133)

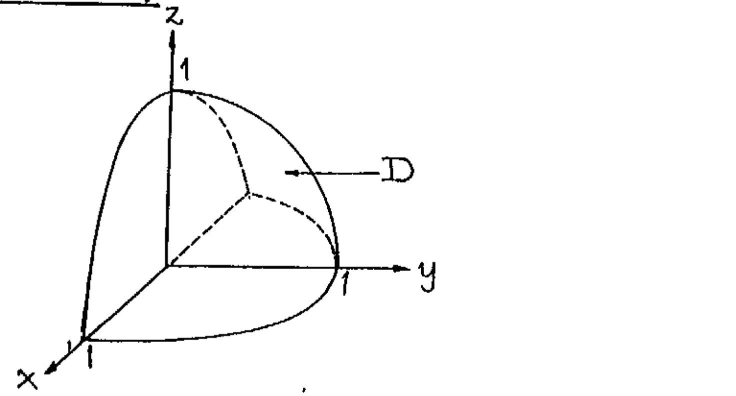
$(x-a)^2+(y-b)^2+(z-c)^2=x^2+y^2+z^2+a^2+b^2+c^2-2ax-2by-2cz$;

$K=\{(x,y,z):\ x^2+y^2+z^2\le 1\}$. $\mathbf{x}=(x,y,z)$, $\mathbf{a}=(a,b,c)$;

$$\iiint_K|\mathbf{x}-\mathbf{a}|^2dxdydz=\iiint_K(x^2+y^2+z^2)\,dxdydz+$$
$$+(a^2+b^2+c^2)\iiint_K dxdydz-2a\iiint_K x\,dxdydz-$$
$$-2b\iiint_K y\,dxdydz-2c\iiint_K z\,dxdydz=\frac{4\pi}{15}+\frac{4\pi}{3}(a^2+b^2+c^2).$$

Anm. Jag har utnyttjat resultaten i Ö.7.8.

Övning 7.10 (s. 133)



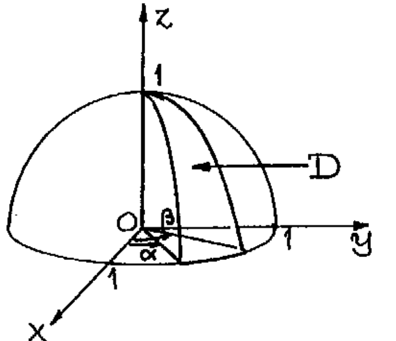
$$\iiint_D z\,dxdydz \left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1 \\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/2 \\ z=r\cos\theta & \underbrace{0\le\phi\le\pi}_{D'} \end{array}\right] =$$

$$=\iiint_{D'} r\cos\theta\, r^2\sin\theta\, dr d\theta d\phi =$$

$$=\int_0^1 r^3 dr \int_0^{\pi/2}\sin\theta\cos\theta\, d\theta\cdot\int_0^{\pi} d\phi =$$

$$=\frac{\pi}{4}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\, d\theta = \frac{\pi}{16}\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{\pi}{8}.$$

Övning 7.11 (s. 133)



$\alpha=\frac{\pi}{4},\ \beta=\frac{\pi}{3}.$

$$\mu(D)=\iiint_D dxdydz \left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1 \\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/2 \\ z=r\cos\theta & \underbrace{\pi/4\le\phi\le\pi/3}_{D'} \end{array}\right] =$$

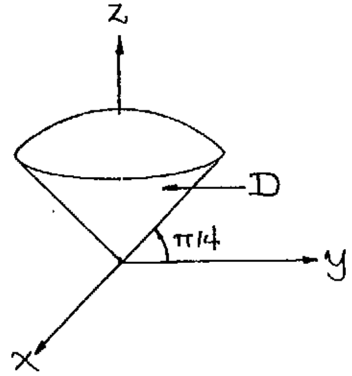
$$=\iiint_{D'} r^2\sin\theta\, dr d\theta d\phi =$$

$$=\int_0^1 r^2 dr\int_0^{\pi/2}\sin\theta\, d\theta\int_{\pi/4}^{\pi/3} d\phi = \frac{1}{3}\cdot 1\cdot\left(\frac{\pi}{3}-\frac{\pi}{4}\right)=\frac{\pi}{36}\ \text{v.e.}$$

Övning 7.12 (s. 133)

$D=\{(x,y,z):\ x^2+y^2+z^2\le 1,\ z\ge\sqrt{x^2+y^2}\}.$

forts.



$$\iiint_D \sqrt{x^2+y^2+z^2}\,dxdydz \left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1 \\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/4 \\ z=r\cos\theta & \underbrace{0\le\phi\le 2\pi}_{D'} \end{array}\right] =$$

$$=\iiint_{D'} r\cdot r^2\sin\theta\, dr d\theta d\phi =$$

$$=\int_0^1 r^3 dr\int_0^{\pi/4}\sin\theta\, d\theta\int_0^{2\pi} d\phi = \frac{1}{4}\left[-\cos\theta\right]_0^{\pi/4}\cdot 2\pi = \frac{\pi}{2}\left(1-\frac{1}{\sqrt{2}}\right)\text{v.e.}$$

Övning 7.13 (s. 133)

$$I(\alpha)=\iiint_D \frac{1}{(x^2+y^2+z^2)^{\alpha/2}}\,dxdydz \left[\begin{array}{l|l} x=r\sin\theta\cos\phi & r\ge 1 \\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi \\ z=r\cos\theta & \underbrace{0\le\phi\le 2\pi}_{D'} \end{array}\right] =$$

$$=\iiint_{D'}\frac{1}{r^\alpha} r^2\sin\theta\, dr d\theta d\phi = \iiint_{D'} r^{2-\alpha}\sin\theta\, dr d\theta d\phi =$$

$$=\int_1^{\infty} r^{2-\alpha} dr\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi} d\phi = 4\pi\cdot\lim_{R\to\infty}\int_1^R r^{2-\alpha} dr =$$

$$=4\pi\lim_{R\to\infty}\left[\frac{1}{3-\alpha}r^{3-\alpha}\right]_1^R = 4\pi\cdot\frac{1}{\alpha-3}\left(1-\lim_{R\to\infty}R^{3-\alpha}\right)=\frac{4\pi}{\alpha-3}.$$

Resultat: Integralen är konvergent för $\alpha>3$; dess värde blir då $\frac{4\pi}{\alpha-3}$.

Anm. Integralens värde är alltid positiv.

Övning 7.14 (s. 133)

$$\iiint_{\mathbb{R}^3} \frac{e^{-r}}{r} dxdydz \left[\begin{array}{l|l} x = r\sin\theta\cos\phi & r \geq 0 \\ y = r\sin\theta\sin\phi & 0 \leq \theta \leq \pi \\ z = r\cos\theta & \underbrace{0 \leq \phi \leq 2\pi}_{D'} \end{array} \right] =$$

$$= \iiint_{D'} \frac{e^{-r}}{r} r^2 \sin\theta \, dr d\theta d\phi =$$

$$= \iiint_{D'} re^{-r} \sin\theta \, dr d\theta d\phi = \int_0^\infty re^{-r} dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi =$$

$$= 4\pi \cdot \lim_{R\to\infty} \int_0^R re^{-r} dr = 4\pi \lim_{R\to\infty} \left[-(r+1)e^{-r}\right]_0^R = 4\pi.$$

Övning 7.15 (s. 133)

$$\mu(K) = \iiint_K \rho(x) dV = \iiint_{\mathbb{R}^3} e^{-(x^2+2y^2+3z^2)} dxdydz =$$

$$= \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-2y^2} dy \int_{\mathbb{R}} e^{-3z^2} dz = \sqrt{\frac{\pi}{1}} \cdot \sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{\pi}{3}} = \sqrt{\frac{\pi^3}{6}}.$$

Övning 7.16 (s. 134)

$$\begin{cases} u = x+2y+2z \\ v = 2x-2y+z \\ w = 2x+y-2z \end{cases} \Rightarrow \frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix} = 27 = \left(\frac{d(x,y,z)}{d(u,v,w)}\right)^{-1};$$

$$\iiint_{\mathbb{R}^3} \frac{e^{-(x+2y+2z)}}{(1+(2x-2y+z)^2)(1+(2x+y-2z)^2)} dxdydz \left[\begin{array}{l} u = x+2y+2z \\ v = 2x-2y+z \\ w = 2x+y-2z \end{array} \right] =$$

$$= \iiint_{\mathbb{R}^3} \frac{e^{-u^2}}{(1+v^2)(1+w^2)} \frac{1}{27} du dv dw = \frac{1}{27} \int_{\mathbb{R}} e^{-u^2} du \int_{\mathbb{R}} \frac{1}{1+v^2} dv \cdot$$

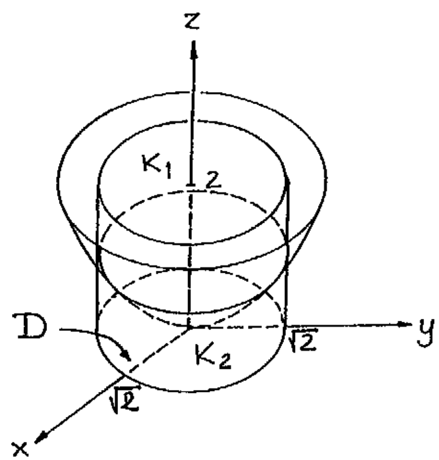
$$\cdot \int_{\mathbb{R}} \frac{1}{1+w^2} dw = \frac{1}{27} \sqrt{\pi} \cdot \pi \cdot \pi = \frac{\pi^{5/2}}{27}.$$

Resultat: Se ovan.

8. Användningar av integraler

Volymberäkningar

Övning 8.1 (s. 140)



$$\mu(K_2) = \iiint_{K_2} dxdydz = \iint_D \left(\int_0^{x^2+y^2} dz \right) dxdy =$$

$$= \iint_D (x^2+y^2) dxdy = \int_0^{\sqrt{2}} r^3 dr \int_0^{2\pi} d\theta = 2\pi \Rightarrow$$

$$\Rightarrow \mu(K_1) = 4\pi - 2\pi = 2\pi = \mu(K_2).$$

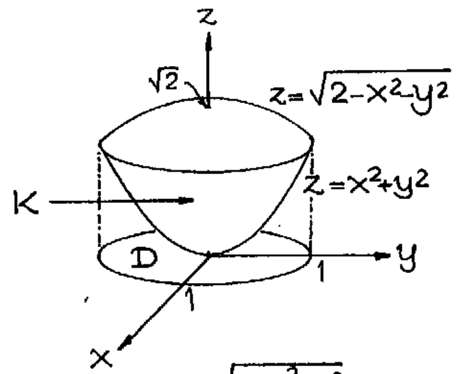
Resultat: Det sökta förhållandet är 1.

Övning 8.2 (s. 140)

$$z - x^2 - y^2 = 0 \Leftrightarrow z = x^2 + y^2 \Rightarrow x^2 + y^2 + z^2 = z^2 + z;$$

$$z^2 + z - 2 = 0 \Leftrightarrow z = 1 \Rightarrow x^2 + y^2 = 1.$$

$$K = \{(x,y,z) : x^2+y^2 \leq z \leq \sqrt{2-x^2-y^2}\}; D = \{(x,y) : x^2+y^2 \leq 1\}.$$



$$\mu(K)=\iiint_K dxdydz=\iint_D\Big(\int_{x^2+y^2}^{\sqrt{2-x^2-y^2}}dz\Big)dxdy=$$

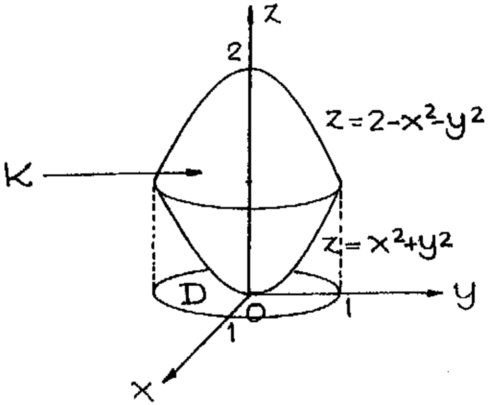
$$=\iint_D\left(\sqrt{2-x^2-y^2}-x^2-y^2\right)dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1\\ y=r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'}\left(\sqrt{2-r^2}-r^2\right)r\,dr\,d\theta=\int_0^1\left(r\sqrt{2-r^2}-r^3\right)dr\int_0^{2\pi}d\theta=$$

$$=\pi\left[-\tfrac{2}{3}(2-r^2)^{3/2}-\tfrac{1}{2}r^4\right]_0^1=\pi\left(-\tfrac{2}{3}-\tfrac{1}{2}+\tfrac{4}{3}\sqrt{2}\right)=\tfrac{\pi}{6}(8\sqrt{2}-7).$$

Resultat: Den sökta volymen är 2,259 ve.

Övning 8.3 (s. 140)



forts.

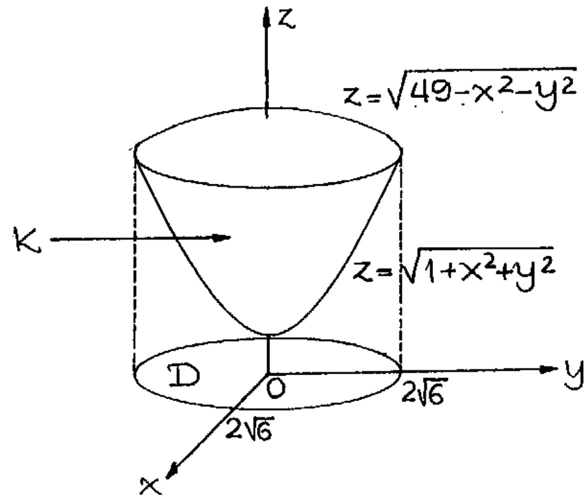
$K=\{(x,y,z): x^2+y^2\le z\le 2-(x^2+y^2)\}$; $D=\{(x,y): x^2+y^2\le 1\}$.

$$\mu(K)=\iiint_K dxdydz=\iint_D\Big(\int_{x^2+y^2}^{2-x^2-y^2}dz\Big)dxdy=$$

$$=\iint_D 2\left(1-(x^2+y^2)\right)dxdy=\int_0^1 2r(1-r^2)dr\int_0^{2\pi}d\theta=$$

$$=4\pi\int_0^1(r-r^3)dr=4\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1=\underline{\pi\ \text{ve.}}$$

Övning 8.4 (s. 140)



$$\sqrt{49-x^2-y^2}=\sqrt{1+x^2+y^2}\Leftrightarrow 49-x^2-y^2=1+x^2+y^2\Leftrightarrow$$

$$\Leftrightarrow 2(x^2+y^2)=48\Leftrightarrow x^2+y^2=24.$$

$K=\{(x,y,z): \sqrt{1+x^2+y^2}\le z\le\sqrt{49-x^2-y^2}\}$.

$D=\{(x,y): x^2+y^2\le 24\}$.

$$\mu(K)=\iiint_K dxdydz=\iint_D\left(\sqrt{49-(x^2+y^2)}-\sqrt{1+x^2+y^2}\right)dxdy=$$

$$=\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 2\sqrt{6} \\ y=r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'} \end{array}\right] = \iint_{D'} (\sqrt{49-r^2}-\sqrt{r^2+1})\, r\,dr\,d\theta =$$

$$= \int_0^{2\sqrt{6}} (\sqrt{49-r^2}-\sqrt{r^2+1})\, r\,dr \cdot 2\pi = 2\pi\left[-\tfrac{1}{3}((49-r^2)^{3/2}-(r^2+1)^{3/2})\right]_0^{2\sqrt{6}}$$

$$= 2\pi\cdot(-\tfrac{1}{3})(5\cdot 25+5\cdot 25-7\cdot 49-1) = \frac{2\pi}{3}\cdot 94 = \frac{188\pi}{3} \approx \underline{196{,}87\text{ ve.}}$$

Övning 8.5 (s. 140)

Vi bestämmer projektionen av området på xy-planet:

$$\begin{cases} z=x^2+y^2 \\ z=2-3x-2y \end{cases} \Rightarrow x^2+y^2=2-3x-2y \Leftrightarrow (x+\tfrac{3}{2})^2+(y+1)^2=\tfrac{21}{4};$$

$K=\{(x,y): x^2+y^2\le z\le 2-3x-2y\}$. (Glöm figuren!)

$D=\{(x,y): (x+\tfrac{3}{2})^2+(y+1)^2\le\tfrac{21}{4}\}$.

$$\mu(K)=\iiint_K dx\,dy\,dz = \iint_D \Big(\int_{x^2+y^2}^{2-3x-2y} dz\Big)\,dx\,dy =$$

$$= \iint_D (2-3x-2y-x^2-y^2)\,dx\,dy \left[\begin{array}{l} x=-\tfrac{3}{2}+r\cos\theta \\ y=-1+r\sin\theta \end{array}\right] =$$

$$= \iint_{D'} (\tfrac{21}{4}-r^2)\, r\,dr\,d\theta =$$

$$= \int_0^{\sqrt{21}/2} (\tfrac{21}{4}r-r^3)\,dr \int_0^{2\pi} d\theta = 2\pi\left[\tfrac{21}{8}r^2-\tfrac{1}{4}r^4\right]_0^{\sqrt{21}/2} =$$

$$= 2\pi\left(\tfrac{21}{8}\cdot\tfrac{21}{4}-\tfrac{1}{4}(\tfrac{21}{4})^2\right) = 2\pi\,\tfrac{1}{4}\cdot(\tfrac{21}{4})^2 = \underline{\frac{441\pi}{32}}.$$

Resultat: Den sökta volymen är 43,295 ve.

Övning 8.6 (s. 140)

$|z|\le x^2+y^2 \Leftrightarrow -(x^2+y^2\le z\le x^2+y^2$.

$x^2+y^2-4x=(x-2)^2+y^2-4\le 0 \Leftrightarrow (x-2)^2+y^2\le 4$.

$K=\{(x,y,z): -(x^2+y^2)\le z\le x^2+y^2,\ (x-2)^2+y^2\le 4\}$.

$D=\{(x,y): (x-2)^2+y^2\le 4\}$.

$$\mu(K)=\iiint_K dx\,dy\,dz = \iint_D \Big(\int_{-(x^2+y^2)}^{x^2+y^2} dz\Big)dx\,dy = \iint_D 2(x^2+y^2)\,dx\,dy =$$

$$= \left[\begin{array}{l|l} x=2+r\cos\theta & 0\le r\le 2 \\ y=r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'} \end{array}\right] = \iint_{D'} 2(r^2+4+4r\cos\theta)\, r\,dr\,d\theta =$$

$$= 2\int_0^2\Big(\int_0^{2\pi}(r^3+4r+4r^2\cos\theta)\,d\theta\Big)dr = 2\int_0^2 (r^3+4r)\,dr\cdot 2\pi =$$

$$= 4\pi\left[\tfrac{1}{4}r^4+2r^2\right]_0^2 = 4\pi(4+8) = \underline{48\pi\text{ ve}}$$

Övning 8.7 (s. 140)

$K=\{(x,y,z): 0\le z\le 10-(x^2+y^2),\ x+1-y^2\ge 0,\ x+y^2-1\le 0\}$

$E=\{(x,y): x+1-y^2\ge 0 \wedge x+y^2-1\le 0\} =$

$= \{(x,y): x\ge y^2-1 \wedge x\le 1-y^2\} =$

$= \{(x,y): -(1-y^2)\le x\le 1-y^2\}$.

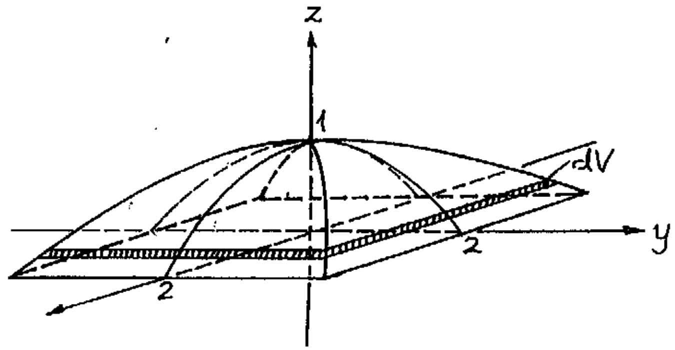
$$\mu(K)=\iiint_K dx\,dy\,dz = \iint_E \Big(\int_0^{10-x^2-y^2} dz\Big)dx\,dy = \iint_E (10-x^2-y^2)\,dx\,dy$$

$$= \int_{-1}^{1}\Big(\int_{y^2-1}^{1-y^2}(10-y^2-x^2)\,dx\Big)dy = \int_{-1}^{1}\Big(\big[(10-y^2)x-\tfrac{1}{3}x^3\big]_{y^2-1}^{1-y^2}\Big)dy$$

$$= \int_{-1}^{1} 2\big((10-y^2)(1-y^2)-\tfrac{1}{3}(1-y^2)^3\big)dy = \int_{-1}^{1} 2\big(\tfrac{29}{3}-10y^2+\tfrac{1}{3}y^6\big)dy$$

$$=2\left[\frac{29}{3}y-\frac{10}{3}y^3+\frac{1}{21}y^7\right]_{-1}^{1}=4\left(\frac{29}{3}-\frac{10}{3}+\frac{1}{21}\right)=\frac{536}{21}\text{ ve.}$$

Övning 8.8 (s.140)



$$\mu(K)=\iiint_K dV=\int_0^1 2x\cdot 2y\cdot dz=4\int_0^1 xy\,dz=16\int_0^1(1-z)dz$$

$$=-\left[16\cdot\tfrac{1}{2}(1-z)^2\right]_0^1=8\text{ ve.}$$

Anm. $x^2=4(1-z)\ \wedge\ y^2=4(1-z)\ \Rightarrow (xy)^2=16(1-z)^2 \Leftrightarrow$

$\Leftrightarrow xy=4(1-z)\Leftrightarrow 4xy=16(1-z)$ (se fig.)

Övning 8.9 (s.140)

$$\mu(K)=\iiint_K dV=\int_{-1}^{1}2x\cdot 2z\,dy=4\int_{-1}^{1}(\sqrt{1-y^2})^2dy=$$

$$=4\int_{-1}^{1}(1-y^2)dy=8\int_0^1(1-y^2)dy=8\cdot\frac{2}{3}=\frac{16}{3}\text{ ve.}$$

Övning 8.10 (s.140)

Vi projicerar kroppen på xy-planet. forts.

$$z^2=2x^2+5y^2=(2x^2+5y^2)^2\Leftrightarrow 2x^2+5y^2=1.$$

$$K=\{(x,y,z):\ \sqrt{2x^2+5y^2}\leq z\leq 2x^2+5y^2\}.$$

$$D=\{(x,y):\ 2x^2+5y^2\leq 1\}.$$

$$\mu(K)=\iiint_K dx\,dy\,dz=\iint_D\Big(\int_{2x^2+5y^2}^{\sqrt{2x^2+5y^2}}dz\Big)dx\,dy=$$

$$=\iint_D\left(\sqrt{2x^2+5y^2}-2x^2-5y^2\right)dx\,dy\left[\begin{array}{l|l}\sqrt{2}x=r\cos\theta & 0\leq r\leq 1\\ \sqrt{5}y=r\sin\theta & \underbrace{0\leq\theta\leq 2\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'}(r-r^2)\frac{1}{\sqrt{10}}r\,dr\,d\theta=\frac{1}{\sqrt{10}}\int_0^1(r^2-r^3)dr\int_0^{2\pi}d\theta=$$

$$=\frac{1}{\sqrt{10}}\left[\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1\cdot 2\pi=\frac{1}{\sqrt{10}}\cdot\frac{1}{12}\cdot 2\pi=\frac{\pi}{6\sqrt{10}}\approx 0{,}166\text{ ve.}$$

Övning 8.11 (s.140)

$$K=\left\{(x,y,z):\ 0\leq z\leq\frac{1}{10}(x+y+100),\ \frac{(x-5)^2}{9}+\frac{(y-7)^2}{4}\leq 1\right\}.$$

$$D=\left\{(x,y):\ \frac{(x-5)^2}{9}+\frac{(y-7)^2}{4}\leq 1\right\}.$$

$$\mu(K)=\iint_D\frac{1}{10}(x+y+100)dx\,dy\left[\begin{array}{l|l}x=5+3r\cos\theta & 0\leq r\leq 1\\ y=7+2\sin\theta & \underbrace{0\leq\theta\leq 2\pi}_{D'}\end{array}\right]=$$

$$=\frac{1}{10}\iint_{D'}(5+3r\cos\theta+7+2r\sin\theta+100)6r\,dr\,d\theta=$$

$$=\frac{3}{5}\iint_{D'}(112+3r\cos\theta+2r\sin\theta)r\,dr\,d\theta=\frac{3}{5}\int_0^1 112r\,dr\cdot 2\pi=$$

$$=\frac{3}{5}\cdot 56\cdot 2\pi=\frac{336\pi}{5}\approx 211{,}115\text{ ve.}$$

Övning 8.12 (s.141)

$$K=\left\{(x,y,z):\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1\right\}$$ ("Sfäriska koordinater")

$$\mu(K)=\iiint_K dV \left[\begin{array}{l|l} x=ar\sin\theta\cos\phi & 0\le r\le 1 \\ y=br\sin\theta\sin\phi & 0\le\theta\le\pi \\ z=cr\cos\theta & 0\le\phi\le 2\pi \end{array}\right]=$$

$$=\iiint_{K'} abcr^2\sin\theta\, dr d\theta d\phi =$$

$$=abc\int_0^1 r^2dr\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = abc\cdot\frac{1}{3}\cdot 2\cdot 2\pi=\frac{4\pi}{3}abc.$$

Tröghetsmoment

Övning 8.13 (S. 141)

$$f(x,y)=\frac{2}{6+x^2+y^2},\quad g(x,y)=\frac{1}{1+x^2+y^2};$$

$$f(x,y)=g(x,y)\Leftrightarrow 6+x^2+y^2=2(1+x^2+y^2)\Leftrightarrow x^2+y^2=4.$$

$$K=\{(x,y,z): \frac{2}{6+x^2+y^2}\le z\le\frac{1}{1+x^2+y^2}\};\ D=\{(x,y): x^2+y^2\le 4\}.$$

$$I=\iiint_K (x^2+y^2)dxdydz=\iint_D (x^2+y^2)\left(\int_{f(x,y)}^{g(x,y)} dz\right)dxdy=$$

$$=\iint_D (x^2+y^2)\left(\frac{1}{1+x^2+y^2}-\frac{2}{6+x^2+y^2}\right)dxdydz=(\text{polärt})=$$

$$=\iint_{D'} r^2\left(\frac{1}{1+r^2}-\frac{2}{6+r^2}\right)rdrd\theta=\int_0^1\left(\frac{12r}{r^2+6}-\frac{r}{r^2+1}-r\right)dr\cdot 2\pi=$$

$$=2\pi\left[6\ln(r^2+6)-\frac{1}{2}\ln(r^2+1)-\frac{1}{2}r^2\right]_0^1=\underline{\pi(11\ln 5-12\ln 3-4)}.$$

Övning 8.14 (S. 141)

$$K=\{(x,y,z): 0\le z\le 1+\sqrt{1-(x^2+y^2)},\ x^2+y^2\le 1\}.$$

$$D=\{(x,y): x^2+y^2\le 1\}.$$

$$I=\iiint_K (x^2+y^2)dxdydz=\iint_D (x^2+y^2)\left(\int_0^{1+\sqrt{1-x^2-y^2}} dz\right)dxdy=$$

$$=\iint_D (x^2+y^2)(1+\sqrt{1-(x^2+y^2)})dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1 \\ y=r\sin\theta & 0\le\theta\le 2\pi\end{array}\right]=$$

$$=\iint_{D'} r^2(1+\sqrt{1-r^2})rdrd\theta=$$

$$=\int_0^1 r^3(1+\sqrt{1-r^2})dr\cdot 2\pi=2\pi\left[\frac{1}{4}r^4-\frac{1}{3}r^2(1-r^2)^{3/2}\right]_0^1+$$

$$+\frac{2\pi}{3}\int_0^1(1-r^2)^{3/2}2rdr=\frac{\pi}{2}-\frac{2\pi}{3}\left[(\sqrt{1-r^2})^5\right]_0^1=\frac{\pi}{2}+\frac{4\pi}{15}=\frac{23}{30}\pi.$$

Övning 8.15 (S. 141)

$$K=\{(x,y,z): x^2+y^2\le z\le 1,\ \frac{1}{4}\le x^2+y^2\le 1\}.$$

$$D=\{(x,y): \frac{1}{4}\le x^2+y^2\le 1\}.$$

$$I=\iiint_K (x^2+y^2)dxdydz=\iint_D\left(\int_{x^2+y^2}^1 dz\right)(x^2+y^2)dxdy=$$

$$=\iint_K (1-x^2-y^2)(x^2+y^2)dxdy\quad(\text{polära koordinater})=$$

$$=\iint_{D'}(1-r^2)r^2rdrd\theta=\int_{1/2}^1(r^3-r^5)dr\,2\pi=2\pi\left[\frac{r^4}{4}-\frac{r^6}{6}\right]_{1/2}^1=$$

$$=2\pi\cdot\left(\frac{1}{4}-\frac{1}{6}-\frac{1}{64}+\frac{1}{384}\right)=\frac{2\pi}{384}\cdot 27=\frac{9\pi}{64}.$$

Övning 8.16 (S. 141)

$$z=6-x^2-2y^2=x^2+y^2\Leftrightarrow 2x^2+3y^2=6\Leftrightarrow\frac{x^2}{3}+\frac{y^2}{2}=1.$$

$$K=\{(x,y,z): x^2+y^2\le z\le 6-x^2-2y^2\};$$

$$D=\{(x,y): \frac{x^2}{3}+\frac{y^2}{2}\le 1\}.$$

$$I=\iiint_K (x^2+y^2)dxdydz=\iint_D\left(\int_{x^2+y^2}^{2-x^2-2y^2} dz\right)(x^2+y^2)dxdy=$$

$$=\iint_D (x^2+y^2)(6-(x^2+y^2))\,dxdy \left[\begin{array}{l|l} x=\sqrt{3}r\cos\theta & 0\le r\le 1\\ y=\sqrt{2}r\sin\theta & 0\le\theta\le 2\pi\end{array}\right]=$$

$$=\iint_{D'} 6(1-r^2)(3r^2\cos^2\theta+2r^2\sin^2\theta)\sqrt{6}\,r\,dr\,d\theta=$$

$$=6\sqrt{6}\int_0^1(r-r^3)\left(\int_0^{2\pi}\left(\tfrac{5}{2}r^2+\tfrac{1}{2}r^2\cos2\theta\right)d\theta\right)dr=$$

$$=6\sqrt{6}\int_0^1(r-r^3)\left(\int_0^{2\pi}\tfrac{5}{2}r^2\,d\theta+\cancelto{0}{\int_0^{2\pi}\tfrac{1}{2}r^2\cos2\theta\,d\theta}\right)dr=$$

$$=6\sqrt{6}\int_0^1(r-r^3)r^2\cdot5\pi=30\sqrt{6}\pi\underbrace{\int_0^1(r^3-r^5)\,dr}_{1/12}=\frac{5\sqrt{6}\pi}{2}.$$

Övning 8.17 (S. 142)

$2x^2+y^2+z^2+2y+4z = 2x^2+(y+1)^2+(z+2)^2-5$;

$K=\left\{(x,y,z):\ \dfrac{x^2}{(\sqrt{5/2})^2}+\dfrac{(y+1)^2}{(\sqrt{5})^2}+\dfrac{(z+4)^2}{(\sqrt{5})^2}\le 1\right\}.$

$$I=\iiint_K (x^2+y^2)\,dxdydz\left[\begin{array}{l|l} x=\sqrt{5/2}\,r\sin\theta\cos\phi & 0\le r\le 1\\ y=-1+\sqrt{5}\,r\sin\theta\sin\phi & 0\le\theta\le\pi\\ z=-4+\sqrt{5}\,r\cos\theta & 0\le\phi\le 2\pi\end{array}\right]=$$

$$=\iiint_{K'}\left(\tfrac{5}{2}r^2\sin^2\theta\cos^2\phi+5r^2\sin^2\theta\sin^2\phi-2\sqrt{5}r\sin\theta\sin\phi+\right.$$

$$\left.+1\right)\cdot5\sqrt{5/2}\,r^2\sin\theta\,dr\,d\theta\,d\phi=$$

$$=\iiint_{K'}5\left(\tfrac{5}{2}\right)^{3/2}r^4\sin^3\theta\cos^2\phi\,dr\,d\theta\,d\phi+$$

$$+\iiint_{K'}25\sqrt{5/2}\,r^4\sin^3\theta\sin^2\phi\,dr\,d\theta\,d\phi-$$

$$-\iiint_{K'}25\sqrt{2}\,r^3\sin^2\theta\sin\phi\,dr\,d\theta\,d\phi+$$

$$+\iiint_{K'}5\sqrt{5/2}\,r^2\sin\theta\,dr\,d\theta\,d\phi=I_1+I_2-I_3+I_4;\ \text{forts.}$$

$$I_1=5\left(\tfrac{5}{2}\right)^{3/2}\int_0^1 r^4\,dr\int_0^\pi\sin^3\theta\,d\theta\int_0^{2\pi}\cos^2\phi\,d\phi=$$

$$=\left(\tfrac{5}{2}\right)^{3/2}\left[r^5\right]_0^1\cdot\int_0^\pi(1-\cos^2\theta)\sin\theta\,d\theta\int_0^{2\pi}\tfrac{1}{2}(1+\cos2\phi)\,d\phi$$

$$=\left(\tfrac{5}{2}\right)^{3/2}\left[-\cos\theta+\tfrac{1}{3}\cos^3\theta\right]_0^\pi\cdot\left[\tfrac{1}{2}(\phi+\tfrac{1}{2}\sin2\phi)\right]_0^{2\pi}=$$

$$=\left(\tfrac{5}{2}\right)^{3/2}\cdot\tfrac{4}{3}\cdot\pi=\tfrac{5}{2}\cdot\tfrac{\sqrt{10}}{2}\cdot\tfrac{4}{3}\cdot\pi=\frac{5\sqrt{10}\pi}{3}.$$

$$I_2=25\sqrt{5/2}\int_0^1 r^4\,dr\int_0^\pi\sin^3\theta\,d\theta\int_0^{2\pi}\sin^2\phi\,d\phi=$$

$$=25\cdot\sqrt{5/2}\cdot\tfrac{1}{5}\cdot\tfrac{4}{3}\cdot\pi=5\,\tfrac{\sqrt{10}}{2}\cdot\tfrac{4}{3}\cdot\pi=\frac{10\sqrt{10}\pi}{3}.$$

$$I_3=25\sqrt{2}\int_0^1 r^3\,dr\int_0^\pi\sin^2\theta\,d\theta\,\cancelto{0}{\int_0^{2\pi}\sin\phi\,d\phi}=0.$$

$$I_4=5\sqrt{5/2}\int_0^1 r^2\,dr\int_0^\pi\sin\theta\,d\theta\int_0^{2\pi}d\phi=5\cdot\tfrac{\sqrt{10}}{2}\cdot\tfrac{1}{3}\cdot2\cdot2\pi=$$

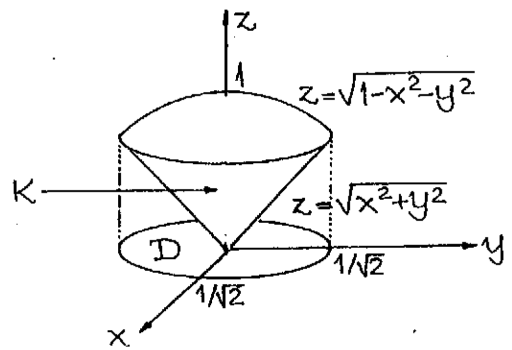
$$=\frac{10\sqrt{10}\pi}{3}.$$

Resultat: Tröghetsmomentet är $\dfrac{25\sqrt{10}\pi}{3}$.

Masscentrum

Övning 8.18 (S. 142)

$K=\{(x,y,z):\ \sqrt{x^2+y^2}\le z\le\sqrt{1-(x^2+y^2)}\}.$



forts.

P.g.a. rotationssymmetrin ligger masscentrum på z-axeln, det vill säga $x_{mc}=y_{mc}=0$.

$$\mu(K)=\iiint_K dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/4\\ z=r\cos\theta & \underbrace{0\le\phi\le 2\pi}_{K'}\end{array}\right]=$$

$$=\iiint_{K'} r^2\sin\theta\,drd\theta d\phi=\int_0^1 r^2dr\int_0^{\pi/4}\sin\theta d\theta\int_0^{2\pi}d\phi=$$

$$=\left[\tfrac{1}{3}r^3\right]_0^1\cdot\left[-\cos\theta\right]_0^{\pi/4}\cdot\left[\phi\right]_0^{2\pi}=\tfrac{1}{3}\left(1-\tfrac{1}{\sqrt{2}}\right)\cdot 2\pi=\tfrac{1}{3}(2-\sqrt{2})\pi\,;$$

$$\mu(K)\,z_{mc}=\iiint_K z\,dxdydz\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le 1\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/4\\ z=r\cos\theta & \underbrace{0\le\phi\le 2\pi}_{K'}\end{array}\right]=$$

$$=\iiint_{K'} r^3\sin\theta\cos\theta\,drd\theta d\phi=\int_0^1 r^3dr\int_0^{\pi/4}\sin\theta\cos\theta d\theta\int_0^{2\pi}d\phi=$$

$$=\left[\tfrac{1}{4}r^4\right]_0^1\cdot\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/4}\cdot\left[\phi\right]_0^{2\pi}=\tfrac{1}{4}\cdot\tfrac{1}{4}\cdot 2\pi=\tfrac{\pi}{8}\,;$$

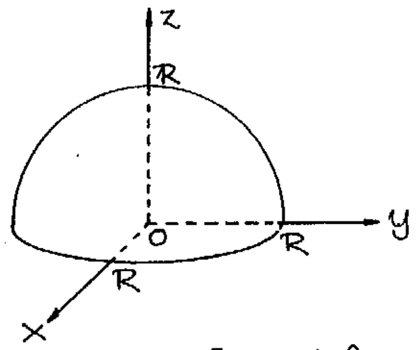
$$\therefore\ \tfrac{1}{3}(2-\sqrt{2})\pi\cdot z_{mc}=\tfrac{1}{8}\pi\Leftrightarrow z_{mc}=\tfrac{3}{8}\,\tfrac{1}{2-\sqrt{2}}=\tfrac{3}{16}(2+\sqrt{2}).$$

<u>Resultat</u>: Kroppens masscentrum har koordinaterna $(0;0;\ \frac{3(2+\sqrt{2})}{16})=(0;0;0{,}640)$.

<u>Övning 8.19 (S.142)</u>

$K=\{(x,y,z):\ x^2+y^2+z^2\le R^2,\ x\ge 0,\ z\ge 0\}$

Kroppen är en kvartssfär (Se fig. på nästa sida), varför $\mu(K)=\frac{\pi}{3}R\,\rho_0$, ρ_0 konstant.



(i) $$\tfrac{\pi}{3}R^3\rho_0\,x_{mc}=\rho_0\iiint_K x\,dV\left[\begin{array}{l|l} x=r\sin\theta\cos\phi & 0\le r\le R\\ y=r\sin\theta\sin\phi & 0\le\theta\le\pi/2\\ z=r\cos\theta & -\pi/2\le\phi\le\pi/2\end{array}\right]=$$

$$=\rho_0\iiint_{K'} r^3\sin^2\theta\cos\phi\,drd\theta d\phi=$$

$$=\rho_0\int_0^R r^3dr\int_0^{\pi/2}\sin^2\theta d\theta\int_{-\pi/2}^{\pi/2}\cos\phi\,d\phi=\rho_0\cdot\tfrac{R^4}{4}\cdot\tfrac{\pi}{4}\cdot 2=\tfrac{\pi}{8}R^4\rho_0$$

$$\Leftrightarrow x_{mc}=\tfrac{\pi}{8}R^4\rho_0\,/\,\tfrac{\pi}{3}R^3\rho_0=\tfrac{3}{8}R.$$

(ii) $$\tfrac{\pi}{3}R^3\rho_0\,y_{mc}=\rho_0\iiint_K y\,dV=\rho_0\iiint_{K'} r^3\sin^2\theta\sin\phi\,drd\theta d\phi=$$

$$=\rho_0\int_0^R r^3dr\int_0^{\pi/2}\sin^2\theta d\theta\underbrace{\int_{-\pi/2}^{\pi/2}\sin\phi\,d\phi}_{0}=0\Leftrightarrow y_{mc}=0.$$

(iii) $$\tfrac{\pi}{3}R^3\rho_0\,z_{mc}=\rho_0\iiint_K z\,dV=\rho_0\iiint_{K'} r^3\sin\theta\cos\theta\,drd\theta d\phi=$$

$$=\rho_0\int_0^R r^3dr\int_0^{\pi/2}\sin\theta\cos\theta d\theta\cdot\int_{-\pi/2}^{\pi/2}d\phi=\rho_0\,\tfrac{1}{4}R^4\,\tfrac{1}{2}\,\pi=\tfrac{\pi}{8}R^4\rho_0.$$

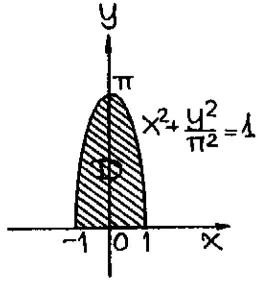
$$\Leftrightarrow z_{mc}=\tfrac{\pi}{8}R^4\rho_0\,/\,\tfrac{\pi}{3}R^3\rho_0=\tfrac{3}{8}R.$$

<u>Resultat</u>: K:s tyngdpunkt ligger i $(\frac{3R}{8}, 0, \frac{3R}{8})$.

<u>Övning 8.20 (S.142)</u>

$D=\{(x,y):\ x^2+\frac{y^2}{\pi^2}\le 1\}$; $\rho(x)=\rho_0=$ konstant.

a) $\mu(D)=\frac{1}{2}\cdot 1\cdot\pi\cdot\pi=\frac{\pi^2}{2}$;



$\mu(D)y_{mc}=\iint_D y\,dxdy=$

$$=\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1 \\ y=\pi r\sin\theta & \underbrace{0\le\theta\le\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'}\pi^2r^2\sin\theta\,drd\theta=\pi^2\int_0^1 r^2dr\int_0^\pi\sin\theta\,d\theta=\frac{2\pi^2}{3}\Leftrightarrow$$

$$\Leftrightarrow\frac{\pi^2}{2}y_{mc}=\frac{2}{3}\pi^2\Leftrightarrow \underline{y_{mc}=\frac{4}{3}}=l.$$

b) $I_l=\iint_D(y-l)^2dxdy=\iint_D(y^2-2ly+l^2)dxdy=$

$$=\iint_D y^2dxdy-2l\iint_D y\,dxdy+l^2\iint_D dxdy=$$

$$=\frac{1}{2}l^2\pi^2-2l\cdot\frac{2}{3}\pi^2+\iint_D y^2dxdy\ [\text{polärt}]=$$

$$=\frac{1}{2}l^2\pi^2-\frac{4l}{3}\pi^2+\int_0^1\pi^3r^3dr\int_0^\pi\sin^2\theta\,d\theta=$$

$$=\frac{1}{2}l^2\pi^2-\frac{4}{3}l\pi^2+\frac{1}{8}\pi^4=\frac{8}{9}\pi^2-\frac{16}{9}\pi^2+\frac{1}{4}\pi^4=\underline{\frac{\pi^4}{4}-\frac{8\pi^2}{9}}.$$

Övning 8.21 (s. 143)

$K=\{(x,y,z):\ y^2\le z\le 2-x^2-y^2\}$.

$y^2=2-x^2-y^2\Leftrightarrow x^2+2y^2=2\Leftrightarrow D=\{(x,y):\ \frac{x^2}{2}+y^2\le 1\}$.

a) $\mu(K)=\iiint_K dxdydz=\iint_D\left(\int_{y^2}^{2-x^2-y^2}dz\right)dxdy=$

$$=\iint_D(2-x^2-2y^2)dxdy\left[\begin{array}{l|l} x=\sqrt{2}r\cos\theta & 0\le r\le 1\\ y=r\sin\theta & 0\le\theta\le 2\pi\end{array}\right]=$$

$$=\iint_{D'}(2-2r^2)r\sqrt{2}\,drd\theta=\sqrt{2}\int_0^1(2r-2r^3)dr\int_0^{2\pi}d\theta=$$

$$=\sqrt{2}\left[r^2-\frac{1}{2}r^4\right]_0^1\cdot 2\pi=\underline{\sqrt{2}\pi}\approx 4{,}443\ \text{ve}.$$

b) $\sqrt{2}\pi\,x_{mc}=\iint_D x(2-(x^2+2y^2))dxdy=(\text{polärt})=$

$$=\iint_{D'}\sqrt{2}r\cos\theta\,(2-2r^2)\sqrt{2}r\,drd\theta=$$

$$=\int_0^1 4(r-r^3)dr\underbrace{\int_0^{2\pi}\cos\theta\,d\theta}_{0}=0\Leftrightarrow x_{mc}=0.$$

$$\sqrt{2}\pi\,y_{mc}=\iint_D y(2-(x^2+2y^2))dxdy=(\text{polärt})=$$

$$=\iint_{D'}2\sqrt{2}(r-r^3)dr\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{0}=0\Leftrightarrow y_{mc}=0.$$

$$\sqrt{2}\pi\,z_{mc}=\iint_D\left(\int_{y^2}^{2-x^2-y^2}z\,dz\right)dxdy=\iint_D\left(\left[\frac{z^2}{2}\right]_{y^2}^{2-x^2-y^2}\right)dxdy=$$

$$=\frac{1}{2}\iint_D((2-x^2-y^2)^2-y^4)dxdy=$$

$$=\frac{1}{2}\iint_D(2-x^2-2y^2)(2-x^2)dxdy\left[\begin{array}{l|l} x=\sqrt{2}r\cos\theta & 0\le r\le 1\\ y=r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'}\end{array}\right]=$$

$$=\frac{1}{2}\cdot 2^2\iint_{D'}(1-r^2)(1-r^2\cos^2\theta)\sqrt{2}r\,drd\theta=$$

$$=2\sqrt{2}\int_0^1(1-r^2)r\left(\int_0^{2\pi}(1-r^2\cos^2\theta)d\theta\right)dr=$$

$$=2\sqrt{2}\int_0^1(r-r^3)(2\pi-\pi r^2)dr=2\sqrt{2}\pi\int_0^1(r^5-3r^3+2r)dr=$$

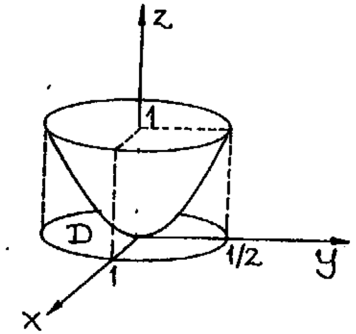
$$=2\sqrt{2}\pi\left[\frac{1}{6}r^6-\frac{3}{4}r^4+r^2\right]_0^1=2\sqrt{2}\pi\cdot\left(\frac{1}{6}-\frac{3}{4}+1\right)=\frac{5\sqrt{2}\pi}{6}\Leftrightarrow$$

$$\Leftrightarrow z_{mc}=\frac{5}{6}\sqrt{2}\pi/\sqrt{2}\pi=\frac{5}{6}.$$

Resultat: a) $\mu(K)=\sqrt{2}\pi$; b) Kroppens tyngdpunkt (masscentrum) ligger $(0,0,\frac{5}{6})$.

Övning 8.22 (s.143)

$K = \{(x,y,z):\ x^2+4y^2 \le z \le 1\}$.



P.g.a. symmetrin ligger tyngdpunkten på z-axel. Det innebär att $x_{mc} = y_{mc} = 0$.

$$\mu(K) = \iiint_K dx\,dy\,dz = \iint_D \Big(\int_{x^2+4y^2}^{1} dz\Big)\,dx\,dy =$$

$$= \iint_D (1-(x^2+4y^2))\,dx\,dy \left[\begin{array}{l|l} x = r\cos\theta & 0\le r\le 1 \\ y = \frac{1}{2}r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'} \end{array}\right] =$$

$$= \iint_{D'} (1-r^2)\cdot\frac{1}{2}r\,dr\,d\theta = \int_0^1 \frac{1}{2}(r-r^3)\,dr \int_0^{2\pi} d\theta =$$

$$= \frac{1}{2}\cdot 2\pi\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{\pi}{4};$$

$$\frac{\pi}{4} z_{mc} = \iiint_K z\,dx\,dy\,dz = \iint_D \Big(\int_{x^2+4y^2}^{1} z\,dz\Big)\,dx\,dy =$$

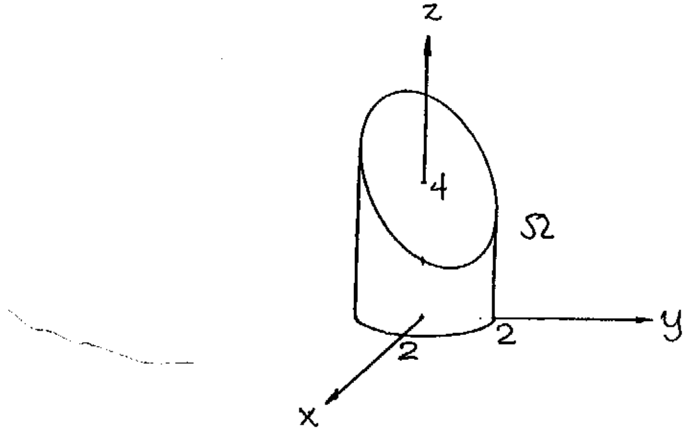
$$= \frac{1}{2}\iint_D (1-(x^2+4y^2)^2)\,dx\,dy \left[\begin{array}{l|l} x = r\cos\theta & 0\le r\le 1 \\ y = \frac{1}{2}r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'} \end{array}\right] =$$

$$= \frac{1}{4}\iint_{D'} (1-r^4)\,r\cdot dr\,d\theta =$$

$$= \frac{1}{4}\left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1 \cdot 2\pi = \frac{\pi}{6} \Leftrightarrow z_{mc} = \frac{2}{3}.\quad x_{mc} = (0,0,\tfrac{2}{3}).$$

Blandade integraltillämpningar

Övning 8.23 (s.144)



$\Omega = \{(x,y,z):\ 0 \le z \le 4-x-y,\ x^2+y^2 \le 4\}$.

$D = \{(x,y):\ x^2+y^2 \le 4\}$.

$$I = \iiint_\Omega (x^2+y^2)\,dx\,dy\,dz = \iint_D (x^2+y^2)\Big(\int_0^{4-x-y} dz\Big)\,dx\,dy =$$

$$= \iint_D (x^2+y^2)(4-x-y)\,dx\,dy \left[\begin{array}{l|l} x = r\cos\theta & 0\le r\le 2 \\ y = r\sin\theta & \underbrace{0\le\theta\le 2\pi}_{D'} \end{array}\right] =$$

$$= \iint_{D'} r^2(4 - r\sin\theta - r\cos\theta)\,r\,dr\,d\theta =$$

$$= \int_0^2 r^3\Big(\int_0^{2\pi}(4 - r\sin\theta - r\cos\theta)\,d\theta\Big)dr = 8\pi\int_0^2 r^3\,dr =$$

$$= 2\pi\left[r^4\right]_0^2 = \underline{\underline{32\pi}}.$$

Resultat: Kroppens tröghetsmoment m.a.p. z-axeln är 32π (lämpliga enheter).

Övning 8.24 (s. 144)

$D = \{(x,y,z): x^2+y^2 \le 1,\ 0 \le z \le 1\}$. $A = \{(x,y): x^2+y^2 \le 1\}$

$$\iiint_D \frac{z-2}{(x^2+y^2+(z-2)^2)^{3/2}}\,dV = \iint_A \Big(\int_0^1 \frac{z-2}{(x^2+y^2+(z-2)^2)^{3/2}}\,dz\Big)dxdy$$

$$= \iint_A \Big(\Big[-\frac{1}{\sqrt{x^2+y^2+(z-2)^2}}\Big]_{z=0}^{z=1}\Big)dxdy =$$

$$= \iint_A \Big(\frac{1}{\sqrt{x^2+y^2+4}} - \frac{1}{\sqrt{x^2+y^2+1}}\Big)dxdy \left[\begin{array}{c|c} x = r\cos\theta & 0 \le r \le 1 \\ y = r\sin\theta & 0 \le \theta \le 2\pi \end{array}\right] =$$

$$= \int_0^1 \Big(\frac{r}{\sqrt{r^2+4}} - \frac{r}{\sqrt{r^2+1}}\Big)dr \int_0^{2\pi} d\theta = 2\pi\Big[\sqrt{r^2+4} - \sqrt{r^2+1}\Big]_0^1 =$$

$$= 2\pi(\sqrt5 - \sqrt2 - 2 + 1) = \underline{2\pi(\sqrt5 - \sqrt2 - 1)}.$$

Övning 8.25 (s. 144)

Den givna kroppen är den del av rotationsparaboloiden som avskärs av sfäroiden.

$$x^2+y^2+2z^2 \le 1 \Leftrightarrow -\frac{1}{\sqrt2}\sqrt{1-x^2-y^2} \le z \le \frac{1}{\sqrt2}\sqrt{1-x^2-y^2}$$

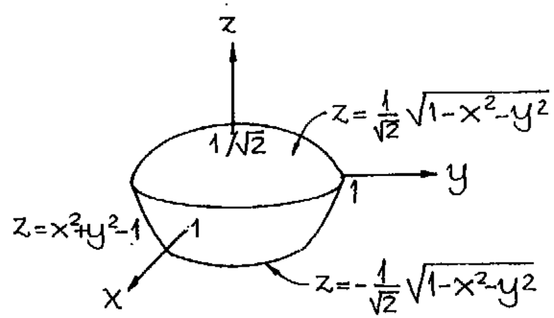
$$K_1 = \{(x,y,z):\ x^2+y^2-1 \le z \le \frac{1}{\sqrt2}\sqrt{1-x^2-y^2}\}$$

$$\begin{cases} z = x^2+y^2-1 \\ x^2+y^2+2z^2 = 1 \end{cases} \Leftrightarrow \begin{cases} z = x^2+y^2-1 \\ z+2z^2 = 0 \end{cases} \Leftrightarrow \begin{cases} z = x^2+y^2-1 \\ z = 0 \vee z = -\frac12 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow x^2+y^2 = 1 \vee x^2+y^2 = \frac12$$

$$K_2 = \{(x,y,z):\ x^2+y^2-1 \le z \le -\frac{1}{\sqrt2}\sqrt{1-x^2-y^2}\}.$$

Den sökta kroppens volym är $\mu(K_1) - \mu(K_2)$.



Det övre "locket" är halva sfäroiden; dess projektion på xy-planet är disken $D_1: x^2+y^2 \le 1$. Det undre "lockets" projektion på xy-planet är disken $D_2: x^2+y^2 \le \frac12$.

$$\mu(K_1) = \iiint_{K_1} dV = \iint_{D_1}\Big(\frac{1}{\sqrt2}\sqrt{1-x^2-y^2} - x^2 - y^2 + 1\Big)dxdy = \text{(polärt)}$$

$$= \int_0^1 \Big(\frac{1}{\sqrt2}\sqrt{1-r^2} - r^2 + 1\Big) r\,dr \int_0^{2\pi} d\theta =$$

$$= 2\pi \int_0^1 \Big(\frac{1}{\sqrt2} r\sqrt{1-r^2} - r^3 + r\Big)dr = 2\pi\Big[-\frac{1}{3\sqrt2}(1-r^2)^{3/2} - \frac{r^4}{4} + \frac{r^2}{2}\Big]_0^1$$

$$= 2\pi\Big(\frac12 - \frac14 + \frac{1}{3\sqrt2}\Big) = \pi\Big(\frac{\sqrt2}{3} + \frac12\Big).$$

$$\mu(K_2) = \iiint_{K_2} dV = \iint_{D_2}\Big(-\frac{1}{\sqrt2}\sqrt{1-x^2-y^2} - x^2 - y^2 + 1\Big)dxdy =$$

$$= \int_0^{1}\Big(1 - r^2 - \frac{1}{\sqrt2}\sqrt{1-r^2}\Big) r\,dr \int_0^{2\pi} d\theta =$$

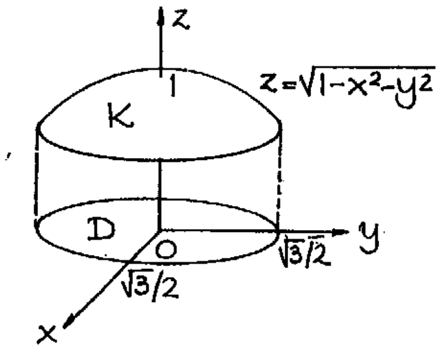
$$= 2\pi \int_0^{1}\Big(r - r^3 - \frac{1}{\sqrt2} r\sqrt{1-r^2}\Big)dr$$

$$= 2\pi\Big[\frac{r^2}{2} - \frac{r^4}{4} - \frac{1}{3\sqrt2}(1-r^2)^{3/2}\Big]_0^{1/\sqrt2} = 2\pi\Big(\frac14 - \frac{1}{16} + \frac{1}{12} - \frac{1}{3\sqrt2}\Big) =$$

$$= 2\pi\Big(\frac{13}{48} - \frac{\sqrt2}{6}\Big) = \pi\Big(\frac{13}{24} - \frac{\sqrt2}{3}\Big)$$

Svar: $\mu(K) = \mu(K_1) - \mu(K_2) = \pi\Big(\frac{2\sqrt2}{3} - \frac{1}{24}\Big) \approx 2{,}831$ ve.

Övning 8.26 (s. 144)



$$K=\{(x,y,z):\ \tfrac{1}{2}\leq z\leq \sqrt{1-x^2-y^2}\},\quad D=\{(x,y):\ x^2+y^2\leq \tfrac{3}{4}\}.$$

$$I=\iiint_K (x^2+y^2)\,dV=\iint_D (x^2+y^2)\Big(\int_{1/2}^{\sqrt{1-x^2-y^2}} dz\Big)dxdy=$$

$$=\iint_D (x^2+y^2)\big(\sqrt{1-x^2-y^2}-\tfrac{1}{2}\big)dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\leq r\leq \frac{\sqrt{3}}{2}\\ y=r\sin\theta & \underbrace{0\leq\theta\leq 2\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'} r^2\big(\sqrt{1-r^2}-\tfrac{1}{2}\big)r\,dr\,d\theta=\int_0^{\sqrt{3}/2}\big(r^3\sqrt{1-r^2}-\tfrac{1}{2}r^3\big)dr\int_0^{2\pi}d\theta=$$

$$=2\pi\Big(\int_0^{\sqrt{3}/2} r^3\sqrt{1-r^2}\,dr\left[\begin{array}{l|l} u^2=1-r^2 & \sqrt{3}/2\to 1/2\\ rdr=-udu & 0\to 1\end{array}\right]-\big[\tfrac{1}{8}r^4\big]_0^{\sqrt{3}/2}\Big)$$

$$=2\pi\Big(\int_1^{1/2}(1-u^2)u\cdot(-udu)-\tfrac{9}{128}\Big)=2\pi\int_{1/2}^{1}(u^2-u^4)\,du-$$

$$-\tfrac{9}{128})=2\pi\big[-\tfrac{1}{5}u^5+\tfrac{1}{3}u^3\big]_{1/2}^{1}-\tfrac{9}{128})=2\pi\big(-\tfrac{1}{5}+\tfrac{1}{3}+\tfrac{1}{160}-\tfrac{1}{24}-$$

$$-\tfrac{9}{128}\big)=2\pi\big(\tfrac{1}{160}-\tfrac{1}{5}+\tfrac{1}{3}-\tfrac{1}{24}-\tfrac{9}{128}\big)=2\pi\big(-\tfrac{31}{160}+\tfrac{7}{24}-\tfrac{9}{128}\big)=$$

$$=\pi\big(\tfrac{7}{12}-\tfrac{31}{80}-\tfrac{9}{64}\big)=(\text{mgn}=960)=\pi\,\frac{7\cdot 80-31\cdot 12-9\cdot 15}{960}=$$

$$=\pi\,\frac{560-372-135}{960}=\frac{53\pi}{960}\approx 0{,}173.$$

Övning 8.27 (s. 144)

$$K_1=\{(x,y,z):\ x^2+y^2-6\leq z\leq 6-2x^2-2y^2\};$$

$$K_2=\{(x,y,z):\ 0\leq z\leq 6-2x^2-2y^2\}.$$

$$x^2+y^2-6=6-2x^2-2y^2\Leftrightarrow x^2+y^2=4\Rightarrow D_1=\{(x,y):x^2+y^2\leq 4\}.$$

(i) $$\mu(K_1)=\iiint_{K_1}dxdydz=\iint_{D_1}\Big(\int_{x^2+y^2-6}^{6-2x^2-2y^2}dz\Big)dxdy=$$

$$=\iint_{D_1}(12-3(x^2+y^2))dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\leq r\leq 2\\ y=r\sin\theta & \underbrace{0\leq\theta\leq 2\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'}(12r-3r^3)\,dr\,d\theta=\int_0^2(12r-3r^3)\,dr\int_0^{2\pi}d\theta=$$

$$=2\pi\big[6r^2-\tfrac{3}{4}r^4\big]_0^2=2\pi(24-12)=24\pi\,;$$

(ii) $$6-2x^2-2y^2=0\Leftrightarrow x^2+y^2=3\Rightarrow D_2=\{(x,y):\ x^2+y^2\leq 3\}.$$

$$\mu(K_2)=\iiint_{K_2}dxdydz=\iint_{D_2}\Big(\int_0^{6-2x^2-2y^2}dz\Big)dxdy=$$

$$=\iint_{D_2}2(3-x^2-y^2)dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\leq r\leq\sqrt{3}\\ y=r\sin\theta & \underbrace{0\leq\theta\leq 2\pi}_{D'}\end{array}\right]=$$

$$=\iint_{D'}2(3-r^2)r\,dr\,d\theta=2\int_0^{\sqrt{3}}(3r-r^3)\,dr\int_0^{2\pi}d\theta=$$

$$=4\pi\big[\tfrac{3}{2}r^2-\tfrac{1}{4}r^4\big]_0^{\sqrt{3}}=4\pi\big(\tfrac{3}{2}\cdot 3-\tfrac{1}{4}\cdot 9\big)=9\pi.$$

<u>Resultat</u>: $\dfrac{\mu(K_2)}{\mu(K_1)}=\dfrac{9\pi}{24\pi}=\dfrac{3}{8}\Leftrightarrow \mu(K_2)=\tfrac{3}{8}\mu(K_1)$.

<u>I ord</u>: Den del av kroppen som ligger ovanför xy-planet har $\frac{3}{8}$ av den totala volymen.

Övning 8.28 (s.144)

a) $\sqrt{|x|}+y^2=t\ge 0 \Leftrightarrow y^2=t-\sqrt{|x|}\ge 0 \Leftrightarrow |x|=t^2 \Leftrightarrow x=\pm t^2$,

$y^2=t-\sqrt{|x|}\ge 0 \Leftrightarrow t\ge\sqrt{|x|} \Leftrightarrow t^2\ge |x| \Leftrightarrow -t^2\le x\le t^2$.

Kurvan är symmetrisk m.a.p. axlarna.

$$A=4\int_0^{t^2}\sqrt{t-\sqrt{x}}\,dx \left[\begin{array}{l|l} u=t-\sqrt{x} & x=t^2\Rightarrow u=0\\ \sqrt{x}=t-u & x=0\Rightarrow u=t\end{array}\right]=$$

$$=4\int_t^0\sqrt{u}\cdot(-2(t-u))\,du=2\cdot 4\int_0^t(t-u)\sqrt{u}\,du=$$

$$=8\int_0^t(t\,u^{1/2}-u^{3/2})\,du=8\left[\tfrac{2}{3}t u^{3/2}-\tfrac{2}{5}u^{5/2}\right]_0^t=$$

$$=8\cdot\left(\tfrac{2}{3}-\tfrac{2}{5}\right)t^{5/2}=\frac{32}{15}t^{5/2}$$ (Är du kvar?)

b) $D=\{(x,y,z):\ \sqrt{|x|+y^2}\le z\le 1\}$, $A=\{(x,y):\ \sqrt{|x|}+y^2\le 1\}$.

$$\mu(D)=\iiint_D d\mu=\iint_A(1-\sqrt{|x|}-y^2)\,dxdy=\mu(A)-$$

$$-\iint_D(\sqrt{|x|}+y^2)\,dxdy=\frac{32}{15}-\int_{-1}^{1}\Big(\int_{-(1-y^2)^2}^{(1-y^2)^2}dx\Big)dy=$$

$$=\frac{32}{15}-\int_{-1}^{1}2(1-y^2)^2\,dy=\frac{32}{15}-2\int_0^1(1-2y^2+y^4)\,dy=$$

$$=\frac{32}{15}-2\left[y-\tfrac{2}{3}y^3+\tfrac{1}{5}y^5\right]_0^1=\frac{32}{15}-2\left(1-\tfrac{2}{3}+\tfrac{1}{5}\right)=$$

$$=\frac{32}{15}-2\cdot\frac{15-10+3}{15}=\frac{32}{15}-\frac{16}{15}=\frac{16}{15}\text{ ve.}$$

$$\frac{16}{15}z_{mc}=\iiint_D z\,dxdydz=\iint_A\Big(\int_{\sqrt{|x|}+y^2}^{1}z\,dz\Big)dxdy=$$

$$=\frac{1}{2}\iint_A\big(1-(\sqrt{|x|}+y^2)^2\big)dxdy=\frac{1}{2}\iint_A dxdy-$$

$$-\frac{1}{2}\iint_A(|x|+y^4+2\sqrt{|x|}y^2)\,dxdy=\frac{1}{2}\cdot\frac{32}{15}-$$

$$-\frac{1}{2}\cdot 2\int_0^1\Big(2\int_0^{(1-y^2)^2}(x+2\sqrt{x}y^2+y^4)\,dx\Big)dy=$$

$$=\frac{16}{15}-2\int_0^1\Big(\Big[x\big(\tfrac{1}{2}x+\tfrac{4}{3}\sqrt{x}y^2+y^4\big)\Big]_0^{(1-y^2)^2}\Big)dy=$$

$$=\frac{16}{15}-2\int_0^1(1-y^2)^2\big(\tfrac{1}{2}(1-y^2)^2+\tfrac{4}{3}(1-y^2)y^2+y^4\big)dy=$$

$$=\frac{16}{15}-2\int_0^1(1-2y^2+y^4)\big(\tfrac{1}{2}-y^2+\tfrac{1}{4}y^4+\tfrac{4}{3}y^2-\tfrac{4}{3}y^4+y^4\big)dy=$$

$$=\frac{16}{15}-2\int_0^1(1-2y^2+y^4)\big(\tfrac{1}{2}+\tfrac{1}{3}y^2-\tfrac{1}{12}y^4\big)dy=$$

$$=\frac{16}{15}-\frac{1}{6}\int_0^1(1-2y^2+y^4)(6+4y^2-y^4)\,dy=$$

$$=\frac{16}{15}-\frac{1}{6}\int_0^1(6-8y^2-3y^4+6y^6-y^8)\,dy=$$

$$=\frac{16}{15}-\frac{1}{6}\big(6-\tfrac{8}{3}-\tfrac{3}{5}+\tfrac{6}{7}-\tfrac{1}{9}\big)=\frac{16}{15}-\frac{1}{6}\big(\tfrac{29}{9}-\tfrac{3}{5}+\tfrac{6}{7}\big)=$$

$$=\frac{16}{15}-\frac{1}{6}\cdot\frac{29\cdot 35-3\cdot 63+6\cdot 45}{9\cdot 5\cdot 7}=\frac{16}{15}-\frac{1}{6}\cdot\frac{1096}{5\cdot 7\cdot 9}=\frac{16}{15}-\frac{1096}{1890}=$$

$$=\frac{920}{1890}=\frac{\cancel{5}\cdot 184}{\cancel{5}\cdot 378}=\frac{2\cdot 92}{2\cdot 189}=\frac{92}{189}\Leftrightarrow z_{mc}=\frac{92}{189}\cdot\frac{15}{16}=\frac{115}{252}.$$

Övning 6.29 (s.145)

$$dm=\rho\,dV\Rightarrow m=\iiint_V\rho\,dV=\iint_S 2(25-h)^2(5-r)2\pi r\,dr\,dh$$

$$=4\pi\int_0^{20}(25-h)^2dh\int_0^4(5r-r^2)\,dr=$$

$$=4\pi\left[-\tfrac{1}{3}(25-h)^3\right]_0^{20}\cdot\left[\tfrac{5}{2}r^2-\tfrac{1}{3}r^3\right]_0^4=4\pi\tfrac{1}{3}(25^3-5^3)\cdot$$

$$\cdot\left(40-\tfrac{64}{3}\right)=\frac{4\pi}{9}(25^3-5^3)(120-64)=\frac{3472\pi}{9}\approx 1{,}2\cdot 10^3\text{ ton.}$$

Övning 6.30 (s.145)

$z=f(x,y)=3-x^2-x^2-y^2$ är en nivåyta till

$F(x,y,z)=3-x^2-y^2-z$;

$\operatorname{grad} F(x,y,z) = (-2x,-2y,-1)$;

$\operatorname{grad} F(1,1,1) = (-2,-2,-1) \Rightarrow \pi_1: 2x+2y+z=5$;

$\operatorname{grad} F(-1,1,1) = (2,-2,-1) \Rightarrow \pi_2: 2x-2y-z=-5$;

$\operatorname{grad} F(1,-1,1) = (-2,2,-1) \Rightarrow \pi_3: 2x-2y+z=5$;

$\operatorname{grad} F(-1,-1,1) = (2,2,-1) \Rightarrow \pi_4: 2x+2y-z=-5$;

$\pi_1 \cap \pi_2 \cap \pi_3 \cap \pi_4 = \{(0,0,5)\}$.

Den sökta volymen är skillnaden mellan volymerna av en pyramid (med sidoytor de 4 planen) och paraboloidens volym ovanför planet. Lägg märke till att pyramiden är kvadratisk och rak.

$$\pi_1 \cap \pi_2 \cap \{x : z=0\} = \{(0, \tfrac{5}{2}, 0)\}.$$

Pyramidens bas har hörnen i $(\pm\frac{5}{2},0,0), (0,\pm\frac{5}{2},0)$.

Dennas volym är således

$$V_1 = \frac{1}{3}\cdot\left(\frac{5}{2}\sqrt{2}\right)^2\cdot 5 = \frac{125}{6}.$$

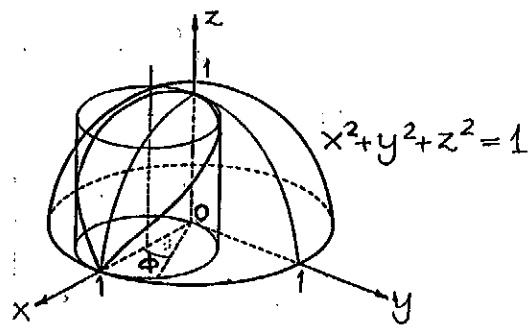
För paraboloiden integrerar vi!

$$V_2 = \iint_{x^2+y^2\le 3} \left(\int_0^{3-x^2-y^2} dz \right) dx\,dy = \iint_{x^2+y^2\le 3} (3-x^2-y^2)\,dx\,dy =$$

$$= \int_0^{\sqrt{3}} r(3-r^2)\,dr \int_0^{2\pi} d\theta = 2\pi\left[\frac{3}{2}r^2 - \frac{1}{4}r^4\right]_0^{\sqrt{3}} = \frac{9\pi}{2}.$$ forts.

Svar: Den sökta volymen är $\frac{125-27\pi}{6} \approx 6{,}696$ ve.

Övning 8.31 (S. 145)



P.g.a. symmetrin räknar vi bara i den 1:a oktanten.

$$\frac{1}{4}V = \iint_D \sqrt{1-x^2-y^2}\,dx\,dy\ ;\quad D = \{(x,y): x^2+y^2 \le x,\ y \ge 0\}$$

Vi inför polära koordinater $r = \cos\phi$, $0 \le \phi \le \frac{\pi}{2}$.

$$\frac{1}{4}V = \int_0^{\pi/2} d\phi \int_0^{\cos\phi} \sqrt{1-r^2}\,r\,dr = \frac{1}{3}\int_0^{\pi/2}(1-\sin^3\phi)\,d\phi =$$

$$= \frac{1}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right) \Leftrightarrow V = \frac{4}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right).$$

V är volymen av den del av cylindern, som omsluts av sfären. Tar man bort den, så finns kvar $\tilde{V} = \frac{4\pi}{3} - V = \frac{4}{3}\left(\frac{\pi}{2} + \frac{2}{3}\right) \approx 2{,}983$ ve.

Anm. Att bestämma V kallas Vivianis problem.

Övning 8.32 (S.145)

a) $f(x,y) = y^2+4x^2-x^4 \Rightarrow \frac{\partial f}{\partial x} = 8x - 4x^3 \wedge \frac{\partial f}{\partial y} = 2y \Rightarrow$

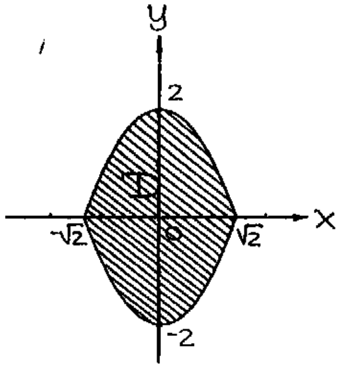
$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 8-12x^2 \wedge \frac{\partial^2 f}{\partial y^2} = 2 \wedge \frac{\partial^2 f}{\partial x \partial y} = 0.$

$Q(h,k) = 2h^2 + 8k^2$ pos. definit $\Rightarrow (0,0)$ minimipunkt.

b) Punkterna $(\pm\sqrt{2},0)$ är också stationära. Enkel kontroll visar att de är sadelpunkter.

Skålens djup är $f(\pm\sqrt{2},0) - f(0,0) = 4$ le.

c) $f(x,y) = 4 \Leftrightarrow y^2 = (x^2-2)^2 \Leftrightarrow \underline{y = x^2-2} \vee \underline{y = 2-x^2}.$



$D = \{(x,y) : x^2-2 \le y \le 2-x^2\}.$

$K = \{(x,y,z) : y^2+4x^2-x^4 \le z \le 4\}.$

$$\mu(K) = \iint_D \left(\int_{y^2+4x^2-x^4}^{4} dz \right) dxdy = \iint_D (4-y^2-4x^2+x^4)\,dxdy =$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{x^2-2}^{2-x^2} (4-y^2-4x^2+x^4)\,dy \right) dx =$$

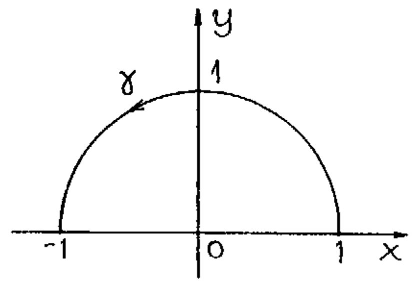
$$= 4\int_0^{\sqrt{2}} \left(\left[4y - \tfrac{1}{3}y^3 + (x^4-4x^2)y \right]_0^{2-x^2} \right) dx =$$

$$= 4\int_0^{\sqrt{2}} \left(\frac{16}{3} - 8x^2 + \frac{1}{3}x^3 + 4x^4 - x^6 \right) dx = \ldots = \frac{1024\sqrt{2}}{105} \text{ve}.$$

9. Vektoranalys i planet

Kurvintegraler

Övning 9.1 (s.154)



$\underline{\gamma(t) = (\cos t, \sin t),\ 0 \le t \le \pi}$ $\quad$ $\underline{\omega = (x^2-y)dx + ydy.}$

$$\begin{cases} x(t) = \cos t \Rightarrow dx = -\sin t\,dt \\ y(t) = \sin t \Rightarrow dy = \cos t\,dt \end{cases} \Rightarrow \omega(\gamma) = (\cos^2 t -$$

$- \sin t)(-\sin t)dt + \sin t \cos t\,dt = (\sin^2 t - \cos^2 t \sin t +$

$+ \sin t \cos t)dt = (\frac{1}{2} - \frac{1}{2}\cos 2t - \cos^2 t \sin t + \sin t \cos t)dt;$

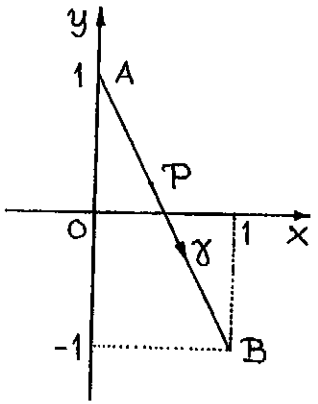
$$\int_\gamma (x^2-y)dx + ydy = \int_0^\pi \omega(\gamma) = \int_0^\pi \frac{1}{2}dt - \frac{1}{2}\int_0^\pi \cos 2t\,dt -$$

$$- \int_0^\pi \cos^2 t \sin t\,dt + \int_0^\pi \sin t \cos t\,dt = \frac{\pi}{2} - 0 - \frac{2}{3} + 0 = \frac{\pi}{2} - \frac{2}{3}.$$

Anm. Att beräkna en kurvintegral är detsamma som att integrera en differentialform. Integrationsvägen är här en sorts oberoende variabel; $\int_\alpha^\beta \omega(\gamma)$ kallas en funktional.

Övning 9.2 (s. 154)

a)



$\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + t\cdot\overline{AB} = \overline{OA} + t(\overline{OB} - \overline{OA}) =$

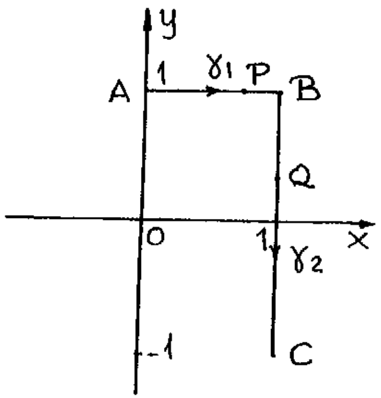
$= (0,1) + t((1,-1) - (0,1)) = (0,1) + t(1,-2) = (x(t), y(t)).$

$\gamma(t) = (t, 1-2t),\ 0 \le t \le 1.$

$\omega = y\,dx - dy = \omega(\gamma) = y(t)\dot{x}(t)dt - \dot{y}(t)dt = (3-2t)dt;$

$\int_\gamma \omega = \int_0^1 \omega(\gamma) = \int_0^1 (3-2t)dt = [3t - t^2]_0^1 = 3 - 1 = 2.$

b)



forts.

$\gamma = \gamma_1 + \gamma_2;$

Jag kommer att använda s som parameter på γ_1 och t som parameter på γ_2.

(i) $\overline{OP} = \overline{OA} + \overline{AP} = \overline{OA} + s\,\overline{AB} = \overline{OA} + s(\overline{OB} - \overline{OA}) =$

$= (0,1) + s((1,1) - (0,1)) = (0,1) + s(1,0) = (s,1);$

$\gamma_1(s) = (x(s), y(s)) = (s,1),\ 0 \le s \le 1.$

$\omega = y\,dx - dy;$

$\omega(\gamma_1) = y(s)\dot{x}(s)ds - \dot{y}(s)ds = 1\cdot 1ds - 0\cdot ds = ds;$

$\int_{\gamma_1} \omega = \int_0^1 \omega(\gamma_1) = \int_0^1 ds = [s]_0^1 = 1.$

(ii) $OQ = OB + OQ = OB + tBC = OB + t(OC - OB) =$

$= (1,1) + t((1,-1) - (1,1)) = (1,1) + t(0,-2) = (1, 1-2t);$

$\gamma_2(t) = (x(t), y(t)) = (1, 1-2t),\ 0 \le t \le 1.$

$\omega = y\,dx - dy;$

$\omega(\gamma_2) = y(t)\dot{x}(t) - \dot{y}(t)dt = (1-2t)\cdot 0dt - (-2)dt = 2dt;$

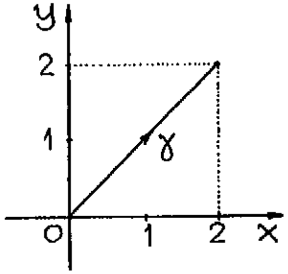
$\int_{\gamma_2} \omega = \int_0^1 \omega(\gamma_2) = \int_0^1 2dt = [2t]_0^1 = 2.$

(iii) $\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega = 1 + 2 = 3.$

Resultat: a) $\int_\gamma \omega = 2$, b) $\int_\gamma \omega = 3$.

Övning 9.3 (s.154)

a)



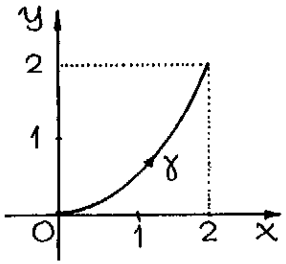
$\gamma(t) = (x(t), y(t)) = (2t, 2t)\ ,\ 0 \leq t \leq 1.$

$\omega = (x^2+xy)dx + (y^2-xy)dy =$

$\omega(\gamma) = (4t^2+2t\cdot 2t)2dt + (4t^2-2t\cdot 2t)dt = 16t^2dt\ ;$

$\int_\gamma \omega = \int_0^1 \omega(\gamma) = \int_0^1 16t^2dt = 16\left[\frac{t^3}{3}\right]_0^1 = \frac{16}{3}.$

b)



$x^2 = 2y\ ;\quad (x = 2t \Rightarrow y = 2t^2)\ ;$

$\gamma(t) = (x(t), y(t)) = (2t, 2t^2)\ ,\ 0 \leq t \leq 1.$

$\omega = (x^2+xy)dx + (y^2-xy)dy\ ;$

$\omega(\gamma) = (4t^2+2t\cdot 2t^2)\cdot 2dt + (4t^4-2t\cdot 2t^2)\cdot 4t\,dt =$

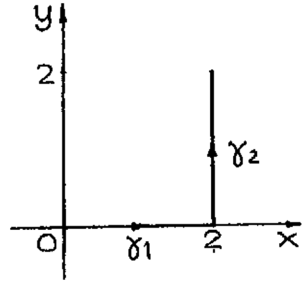
$= (4t^2+8t^3)\cdot 2dt + (4t^4-4t^3)2t\,dt =$

$= (2(4t^2+8t)+2t(4t^4-4t^3))dt = 8(t^2+t^3-2t^4+2t^5)dt\ ;$

$\int_\gamma \omega = \int_0^1 \omega(\gamma) = 8\int_0^1 (t^2+t^3-2t^4+2t^5)dt =$

$= \left[8\left(\frac{t^3}{3}+\frac{t^4}{4}-\frac{2t^5}{5}+\frac{t^6}{3}\right)\right]_0^1 = 8\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{4}+\frac{1}{3}\right) = \frac{62}{15}.$

c)



$\gamma = \gamma_1 + \gamma_2\ ;\quad \omega = (x^2+xy)dx + (y^2-xy)dy.$

(i) $\gamma_1(s) = (s, 0)\ ,\ 0 \leq s \leq 2\ ;$

$\omega(\gamma_1) = s^2ds \Rightarrow \int_{\gamma_1} \omega = \int_0^2 s^2ds = \left[\frac{s^3}{3}\right]_0^2 = \frac{8}{3}.$

(ii) $\gamma_2(t) = (2, t),\ 0 \leq t \leq 2\ ;$

$\omega(\gamma_2) = (4+2t)0dt + (t^2-2t)\cdot 1\,dt = (t^2-2t)dt\ ;$

$\int_{\gamma_2} \omega = \int_0^2 \omega(\gamma_2) = \int_0^2 (t^2-2t)dt = \left[\frac{1}{3}t^3 - t^2\right]_0^2 = -\frac{4}{3}.$

(iii) $\int_\gamma \omega = \left(\int_{\gamma_1} + \int_{\gamma_2}\right)\omega = \frac{8}{3} - \frac{4}{3} = \frac{4}{3}.$

Övning 9.4 (s.154)

$\omega = y \ln\frac{x^2}{y}dx - \frac{x}{y}dy;\ \gamma(t) = (t, t^2),\ \ 1 \leq t \leq 2.$

$\omega(\gamma) = t^2 \ln\frac{t^2}{t^2}dt - \frac{t}{t^2}2t\,dt = -2dt \Rightarrow \int_\gamma \omega = -2\int_1^2 dt = -2.$

Övning 9.5 (s.154)

$\mathbb{F}(x,y)=(1,-2)$;

$\mathbb{r}(t)=(0,1)+t((2,2)-(0,1))=(0,1)+t(2,1)=(2t,1+t)$

$\mathbb{F}\cdot d\mathbb{r}=\mathbb{F}\cdot\dot{\mathbb{r}}\,dt=(1,-2)\cdot(2,1)\,dt=0$

$W=\int_0^1 \mathbb{F}(\mathbb{r}(t))\dot{\mathbb{r}}(t)\,dt=0.$

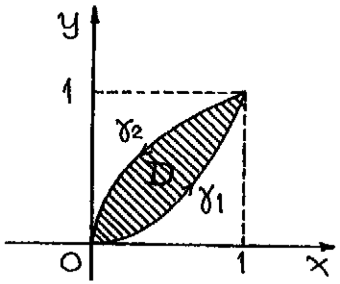
Anm. Kraften är vinkelrät mot vägen.

Övning 9.6 (s.154)

Gradienten är alltid vinkelrät mot nivåkurvor. Om $d\mathbb{r}$ är en liten förflyttning längs γ, så är $\operatorname{grad} f(\mathbb{r})\cdot d\mathbb{r}=0$, så $\operatorname{grad} f$ uträttar inget arbete på partikeln.

Greens formel

Övning 9.7 (s.155)



$\omega=(\underbrace{2xy-x^2+y^2\sin xy^2}_{P(x,y)})dx+(\underbrace{x+y^2+2xy\sin xy^2}_{Q(x,y)})dy$

$d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=(1-2x)dxdy$;

$D=\{(x,y):\ y^2\le x\le\sqrt{y}\}$, $\quad\partial D=\gamma_1+\gamma_2$;

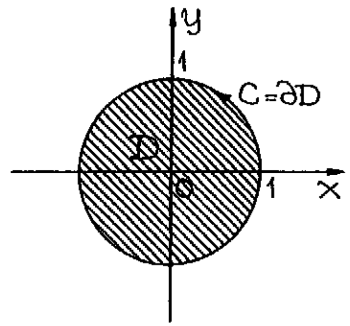
$\oint_{\partial D}\omega=\iint_D d\omega=\iint_D(1-2x)dxdy=\int_0^1\left(\int_{y^2}^{\sqrt{y}}(1-2x)dx\right)dy=$

$=\int_0^1\left([x-x^2]_{y^2}^{\sqrt{y}}\right)dy=\int_0^1(\sqrt{y}-y-y^2+y^4)dy=$

$=\left[\frac{2}{3}y\sqrt{y}-\frac{1}{2}y^2-\frac{1}{3}y^3+\frac{1}{5}y^5\right]_0^1=\frac{2}{3}-\frac{1}{2}-\frac{1}{3}+\frac{1}{5}=\frac{1}{3}+\frac{1}{5}-\frac{1}{2}=\frac{1}{30}.$

Övning 9.8 (s.155)

$\omega=(\underbrace{e^{\sin x}-x^2y}_{P(x,y)})dx+\underbrace{e^{y^2}}_{Q(x,y)}dy\Rightarrow d\omega=\underbrace{\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}_{x^2}dxdy$;



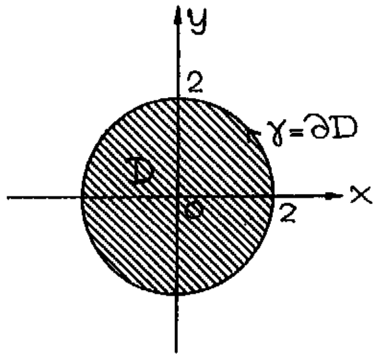
$\oint_C\omega=\iint_D d\omega=\iint_D x^2dxdy\left[\begin{array}{l|l} x=r\cos\theta & 0\le r\le 1\\ y=r\sin\theta & 0\le\theta\le 2\pi\end{array}\right]=$

$=\int_0^1 r^3dr\int_0^{2\pi}\cos^2\theta\,d\theta=\int_0^1 r^3dr\int_0^{2\pi}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta=$

$=\left[\frac{1}{4}r^4\right]_0^1\cdot\left[\frac{\theta}{2}-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}=\frac{1}{4}\cdot\pi=\frac{\pi}{4}.$

Övning 9.9 (s. 155)

$$\omega = \underbrace{(x^3 - x^2y)}_{P(x,y)}dx + \underbrace{xy^2}_{Q(x,y)}dy \Rightarrow d\omega = \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{x^2+y^2}dxdy \;;$$



$$\oint_\gamma \omega = \oint_{\partial D}\omega = \iint_D d\omega = \iint_D (x^2+y^2)dxdy \begin{bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{bmatrix} =$$

$$= \int_0^2 r^3 dr \int_0^{2\pi} d\theta = \left[\tfrac{1}{4}r^4\right]_0^2 \cdot [\theta]_0^{2\pi} = 4\cdot 2\pi = 8\pi.$$

Övning 9.10 (s. 155)

$$\omega = y^2dx + x^2dy \Rightarrow d\omega = \underbrace{\left(\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right)}_{2(x-y)}dxdy \;;$$

$D = \{(x,y) : (x-a)^2 + (y-b)^2 \leq r^2\}$; $\gamma = \partial D$.

$$\oint_\gamma \omega = \oint_{\partial D}\omega = \iint_D 2(x-y)dxdy \begin{bmatrix} x = a + t\cdot\cos\theta \\ y = b + t\sin\theta \end{bmatrix} =$$

$$= \int_0^r \left(\int_0^{2\pi} 2(a + t\cos\theta - b - t\sin\theta)d\theta\right)tdt =$$

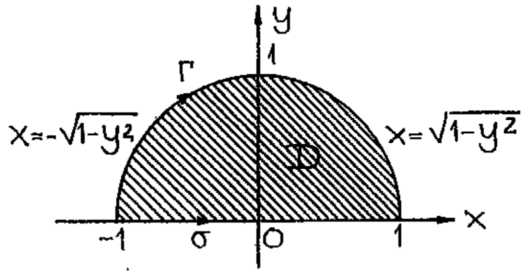
$$= \int_0^r \left(\int_0^{2\pi} 2(a-b)d\theta\right)tdt = 4\pi(a-b)\int_0^r tdt =$$

$$= 2\pi(a-b)[t^2]_0^r = \underline{2\pi(a-b)r^2}.$$

Övning 9.11 (s. 155)

$$\omega = (x^2 - y + 2\ln(1+y))dx + \frac{(1+x)^2}{1+y}dy \;;$$

$D = \{(x,y) : x^2+y^2 \leq 1,\ y \geq 0\}$, $\partial D = -\Gamma + \sigma$.



$$d\omega = \left(\frac{\partial}{\partial x}\frac{(1+x)^2}{1+y} - \frac{\partial}{\partial y}(x^2 - y + 2\ln(1+y))\right)dxdy = \left(1 - \frac{2x}{1+y}\right)dxdy.$$

$$\oint_{\partial D}\omega = \int_{-\Gamma+\sigma}\omega = \int_{-\Gamma}\omega + \int_\sigma \omega = -\int_\Gamma \omega + \int_\sigma \omega = \iint_D d\omega \Leftrightarrow$$

$$\Leftrightarrow \int_\Gamma \omega = \int_\sigma \omega - \iint_D d\omega = \int_{-1}^1 \omega(\sigma) - \iint_D \left(1 + \frac{2x}{y+1}\right)dxdy \;;$$

$\sigma(t) = (t,0)$, $-1 \leq t \leq 1$, $\Rightarrow \omega(\sigma) = t^2dt \Rightarrow \int_\sigma \omega = \frac{2}{3}$;

$$\int_\Gamma \omega = \frac{2}{3} - \iint_D\left(1 + \frac{2x}{y+1}\right)dxdy = \frac{2}{3} - \int_0^1\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\left(1 + \frac{2x}{y+1}\right)dx\right)dy =$$

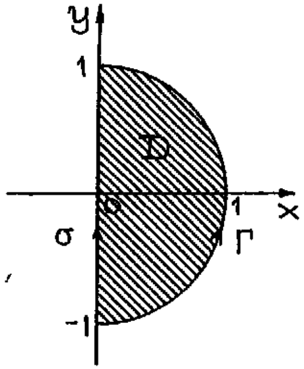
$$= \frac{2}{3} - \int_0^1\left(\left[x + \frac{x^2}{y+1}\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\right)dy = \frac{2}{3} - 2\int_0^1\sqrt{1-y^2}\,dy =$$

$$= \frac{2}{3} - 2\cdot\frac{\pi}{4} = \frac{2}{3} - \frac{\pi}{2}.$$

Övning 9.12 (s. 155)

$$\omega = \mathbb{F}\cdot d\mathbb{r} = (e^x, 1+xy^2)\cdot(dx,dy) = e^xdx + (1+xy^2)dy$$

$$d\omega = \left(\frac{\partial}{\partial x}(1+xy^2) - \frac{\partial}{\partial y}e^x\right)dxdy = y^2dxdy \;;$$



$D = \{(x,y) : x^2+y^2 \leq 1,\ x \geq 0\}$; $\partial D = \Gamma - \sigma$.

$$\oint_{\partial D} \omega = \int_\Gamma \omega - \int_\sigma \omega = \iint_D d\omega \Leftrightarrow \int_\Gamma \omega = \int_\sigma \omega + \iint_D d\omega \ ;$$

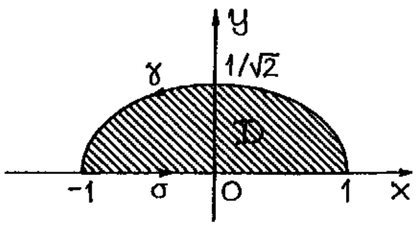
$\sigma(t) = (0,t)$, $-1 \leq t \leq 1$, $\Rightarrow \omega(\sigma) = dt \Rightarrow \int_\sigma \omega = 2$.

$$\int_\Gamma \omega = 2 + \iint_D y^2 dxdy \left[\begin{array}{l|l} x = r\cos\theta & 0 \leq r \leq 1 \\ y = r\sin\theta & -\pi/2 \leq \theta \leq \pi/2 \end{array}\right] =$$

$$= 2 + \int_0^1 r^3 dr \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2 + \frac{1}{4} \cdot \frac{\pi}{2} = 2 + \frac{\pi}{8}.$$

Resultat: $\int_\Gamma \mathbb{F} \cdot d\mathbb{r} = 2 + \frac{\pi}{8}$.

Övning 6.13 (s. 155)



$\omega = (x-y)dx + (x+y)dy \Rightarrow d\omega = 2dxdy$;

$D = \{(x,y) : x^2 + 2y^2 \leq 1,\ y \geq 0\}$. $\partial D = \gamma + \sigma$

$$\oint_{\partial D} \omega = \int_\gamma \omega + \int_\sigma \omega = \iint_D d\omega \Leftrightarrow \int_\gamma \omega = 2\iint_D dxdy - \int_\sigma \omega.$$

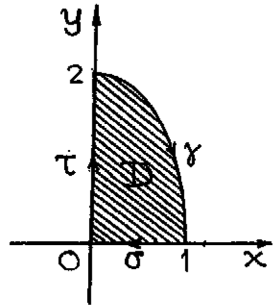
$$2\iint_D dxdy = \pi \cdot 1 \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{2}\pi}{2} \quad (\text{ellipsarean} = \pi \cdot a \cdot b).$$

$\sigma(t) = (t,0)$, $-1 \leq t \leq 1$, $\Rightarrow \omega(\sigma) = t\,dt \Rightarrow \int_\sigma \omega = \int_{-1}^1 t\,dt = 0$.

Resultat: $\int_\gamma (x-y)dx + (x+y)dy = \frac{\pi}{2}\sqrt{2}$.

Övning 9.14 (s. 155)

I.



$\omega = \mathbb{F} \cdot d\mathbb{r} = (y^3, x^3) \cdot (dx, dy) = y^3 dx + x^3 dy$

$D = \{(x,y) : x^2 + \frac{y^2}{4} \leq 1,\ x \geq 0, y \geq 0\}$, $\partial D = \gamma + \sigma + \tau$;

$$\oint_{\partial D} \omega = -\oint_{\partial D} \omega = -\left(\int_\gamma \omega + \int_\sigma \omega + \int_\tau \omega\right) = \iint_D d\omega \Leftrightarrow$$

$$\Leftrightarrow \int_\gamma \omega = -\int_\sigma \omega - \int_\tau \omega - \iint_D d\omega \ ;$$

(i) $-\int_\sigma \omega = \int_{-\sigma} \omega = \int_0^1 \omega(-\sigma) = 0$;

(ii) $\int_\tau \omega = 0$;

(iii) $d\omega = \left(\frac{\partial}{\partial x} x^3 - \frac{\partial}{\partial y} y^3\right) dxdy = 3(x^2 - y^2) dxdy \Rightarrow$

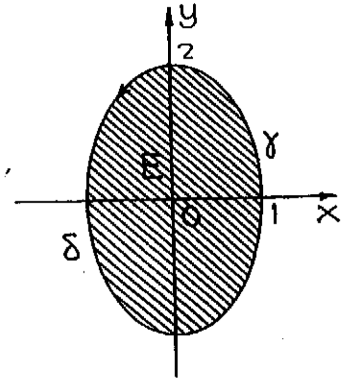
$$\Rightarrow \iint_D 3(x^2-y^2) dxdy \left[\begin{array}{l|l} x = r\cos\theta & 0 \leq r \leq 1 \\ y = 2r\sin\theta & \underbrace{0 \leq \theta \leq \pi/2}_{D'} \end{array}\right] =$$

$$= 6\iint_{D'} r^3(\cos^2\theta - 4\sin^2\theta)\, dr d\theta =$$

forts.

$$=6\int_0^1 r^3\,dr\int_0^{\pi/2}\left(\frac{5}{2}\cos 2\theta-\frac{3}{2}\right)d\theta=6\cdot\frac{1}{4}\cdot\left(-\frac{3}{2}\right)\cdot\frac{\pi}{2}=-\frac{9\pi}{8};$$

II.



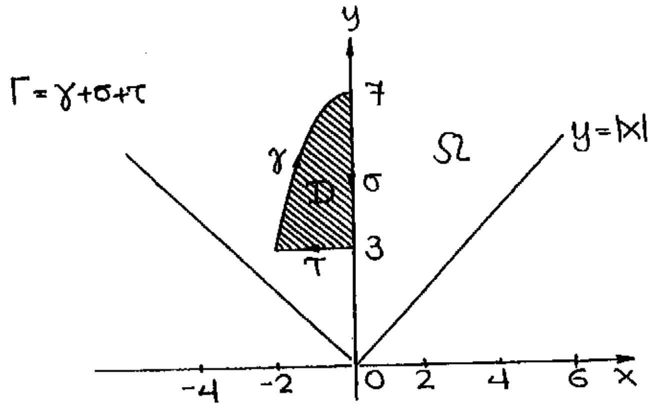
$E=\{(x,y):\ x^2+\frac{y^2}{4}\leq 1\}$; $\partial E=\gamma+\delta$;

γ är kvartsbågen i första hälften av försöket.

$$\oint_{\partial E}\omega=\iint_E d\omega=\iint_E 3(x^2-y^2)\,dx\,dy=6\cdot\frac{1}{4}\cdot\left(-\frac{3}{2}\right)\cdot 2\pi=$$
$$=-\frac{36}{8}\pi \Leftrightarrow \int_\gamma\omega+\int_\delta\omega=-\frac{36}{8}\pi \Leftrightarrow \int_\delta\omega=-\frac{27}{8}\pi.$$

Resultat: Kraften uträttar arbetet $-\frac{27}{8}\pi$.

Övning 9.15 (s.155)



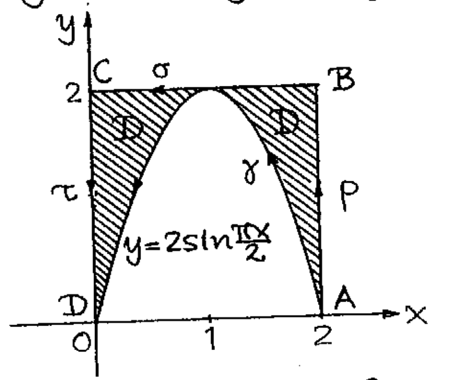
forts.

$\Omega=\{(x,y):\ y>|x|\}\supset D.$

$$\omega=\frac{-x}{(x^2-y^2)^2}dx+\frac{y}{(x^2-y^2)^2}dy\in C^1(\Omega)\Rightarrow d\omega=0\Rightarrow$$
$$\Rightarrow \oint_{\partial D}\omega=\iint_D d\omega=0\Leftrightarrow\int_\gamma\omega+\int_\sigma\omega+\int_\tau\omega=0\Leftrightarrow$$
$$\Leftrightarrow\int_\gamma\omega=-\int_\sigma\omega-\int_\tau\omega=-\int_7^3\frac{1}{y^3}dy-\int_0^{-2}\frac{x}{(x^2-9)^2}dx=\frac{1}{2}\left[\frac{1}{y^2}\right]_7^3+$$
$$+\frac{1}{2}\left[\frac{1}{x^2-9}\right]_{-2}^0=\frac{1}{2}\left(\frac{1}{9}-\frac{1}{49}-\frac{1}{9}+\frac{1}{5}\right)=\frac{1}{2}\left(\frac{1}{5}-\frac{1}{49}\right)=\frac{1}{2}\cdot\frac{44}{245}=\frac{22}{245}.$$

Övning 9.16 (s. 156)

$$\omega=\frac{x^2+y^2-2}{x^2+y^2-2x-2y+2}dx+\frac{4y-x^2-y^2-2}{x^2+y^2-2x-2y+2}dy$$



$$\begin{cases} P(x,y)=\dfrac{x^2+y^2-2}{(x-1)^2+(y-1)^2}\Rightarrow\dfrac{\partial P}{\partial y}=2\cdot\dfrac{x^2-y^2-2xy+2y-2}{((x-1)^2+(y-1)^2)^2}\\ Q(x,y)=\dfrac{4y-x^2-y^2-2}{(x-1)^2+(y-1)^2}\Rightarrow\dfrac{\partial Q}{\partial x}=2\cdot\dfrac{x^2-y^2-2xy+2y-2}{((x-1)^2+(y-1)^2)^2}.\end{cases}\Rightarrow$$

$\Rightarrow d\omega=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=0\Rightarrow\omega$ exakt $\Rightarrow\int_\gamma\omega$ oberoende av γ. Vi tar oss från A till via polygonen, för det är möjligt att räkna ut $\int_\gamma\omega$.

(i) $\int_\rho \omega = \int_1^2 \omega(\rho) = \int_1^2 \frac{4y-4-y^2-2}{(y-1)^2+1}dy = -\int_0^2 \frac{y^2-4y+6}{(y-1)^2+1}dy =$

$= \int_0^2 \left(-1+\frac{2(y-1)}{(y-1)^2+1} - \frac{2}{(y-1)^2+1}\right)dy =$

$= \left[\ln(y^2-2y+2) - 2\arctan(y-1) - y\right]_0^2 = -\pi-2.$

(ii) $\int_\sigma \omega = \int_2^0 \frac{x^2+2}{(x-1)^2+1}dx = \int_2^0 \left(1+\frac{2x-2}{(x-1)^2+1} + \frac{2}{(x-1)^2+1}\right)dx =$

$= \left[x+\ln(x^2-2x+2)+2\arctan(x-1)\right]_2^0 = -\pi-2.$

(iii) $\int_\tau \omega = \int_2^0 \omega(\tau) = \int_2^0 \frac{4y-y^2-2}{(y-1)^2+1}dy = \int_0^2 \frac{y^2-4y+2}{(y-1)^2+1}dy =$

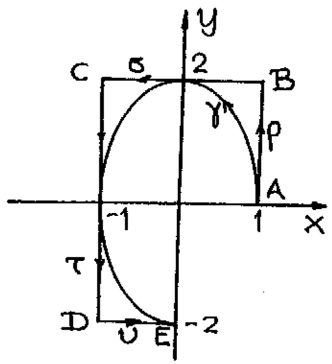
$= \int_0^2 \left(1-\frac{2y-2}{(y-1)^2+1} - \frac{2}{(y-1)^2+1}\right)dy =$

$= \left[y-\ln(y^2-2y+2)-2\arctan(y-1)\right]_0^2 = 2-\pi$;

(i)–(iii) $\Rightarrow \int_\gamma \omega = -\pi-2-\pi-2+2-\pi = -3\pi-2$

Anm. Vi kan inte välja segmentet AD, ty detta och kurvbågen omsluter (1,1), som är singulär för ω. Se Anmärkning på s. 299.

Övning 9.17



forts.

$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \Rightarrow d\omega = 0 \Rightarrow \omega$ exakt $\Rightarrow$

$\Rightarrow \int_\gamma \omega$ oberoende av vägen. Vi tar oss från A till E via polygonen ABCDE.

$\int_\gamma \omega = \int_\rho \omega + \int_\sigma \omega + \int_\tau \omega + \int_\upsilon \omega$ (υ står för ýpsilon)

(i) $\int_\rho \omega = \int_0^2 \omega(\rho) = \int_0^2 \frac{1}{y^2+1}dy = [\arctan y]_0^2 = \arctan 2$;

(ii) $\int_\sigma \omega = \int_{+1}^{-1} \omega(\sigma) = \int_1^{-1} -\frac{2}{x^2+4}dx = 4\int_0^1 \frac{1}{x^2+4}dx = 2\arctan\frac{1}{2}$;

(iii) $\int_\tau \omega = \int_2^{-2} \omega(\tau) = \int_2^{-2} -\frac{1}{y^2+1}dy = 2\int_0^2 \frac{1}{y^2+1} = 2\arctan 2$;

(iv) $\int_\upsilon \omega = \int_{-1}^0 \omega(\upsilon) = \int_{-1}^0 \frac{2}{x^2+4}dx = \arctan\frac{1}{2}$;

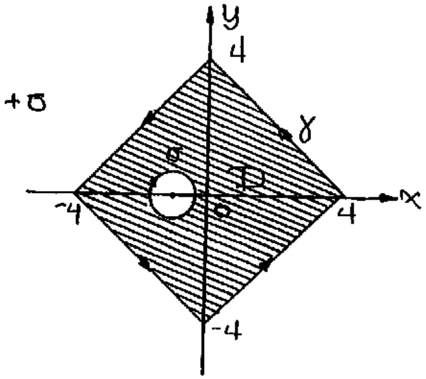
(i)–(iv) $\Rightarrow \int_\gamma \omega = 3\arctan 2 + 3\arctan\frac{1}{2} = 3\cdot\frac{\pi}{2} = \frac{3\pi}{2}$.

Anm. $x>0 \Rightarrow \arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}$.

Övning 9.18 (s. 156)

$\omega = \frac{y}{(x+1)^2+y^2}dx - \frac{x+1}{(x+1)^2+y^2}dy$;

$\partial D = \gamma + \sigma$



forts.

$D=\{(x,y): |x|+|y|\le 4,\ (x+1)^2+y^2\ge \varepsilon^2,\ \varepsilon<1\}$.

$\forall x\in D: d\omega=(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=0 \Rightarrow \oint_\gamma \omega+\int_\sigma \omega=$

$=\iint_D d\omega=0 \Leftrightarrow \oint_\gamma \omega=-\int_\sigma \omega=\int_{-\sigma}\omega=-\int_0^{2\pi}d\theta=-2\pi$

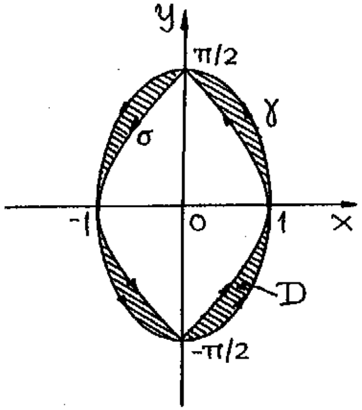
Övning 9.19 (s. 156)

$\omega=-\frac{\sin y}{x^2+\sin^2 y}dx+\frac{x\cos y}{x^2+\sin^2 y}dy\in C^1(\mathbb{R}^2\setminus\{(0,0)\})$

$d\omega=(\frac{\sin^2 y\cos y-x^2\cos y}{(x^2+\sin^2 y)^2}-\frac{\sin^2 y\cos y-x^2\cos y}{(x^2+\sin^2 y)^2})dxdy=0$

$\Rightarrow \int_\gamma \omega$ oberoende av vägen.

$D=\{(x,y): x^2+\frac{4y^2}{\pi^2}\le 1\}\setminus\{(x,y): |x|\le \cos y\}$.



$\oint_{\partial D}\omega=\int_\gamma \omega+\int_{-\sigma}\omega=\iint_D d\omega=0 \Leftrightarrow \int_\gamma \omega=\int_\sigma \omega=\int \omega(\sigma)=$

$=\int_{x=\cos y}\omega+\int_{x=-\cos y}\omega=2\int_{-\pi/2}^{\pi/2}(\frac{\sin^2 y}{\cos^2 y+\sin^2 y}+\frac{\cos^2 y}{\cos^2 y+\sin^2 y})dy=$

$=2\int_{-\pi/2}^{\pi/2}dy=2\pi$

Anm. σ i figuren är negativt orienterad.

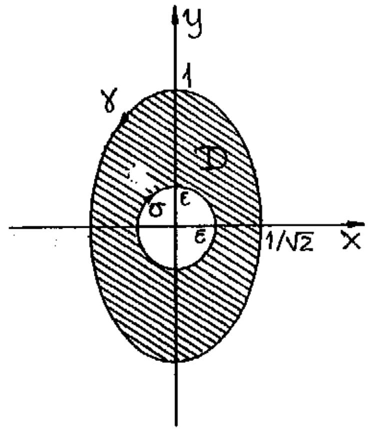
Övning 9.20 (s. 156)

a) $\omega=Pdx+Qdy$ säges vara en exakt differential(form) om det existerar $u\in C^2(\Omega)$, Ω öppet sammanhängande område i $\mathbb{R}^2$, s.a. $P=\frac{\partial u}{\partial x}$ och $Q=\frac{\partial u}{\partial y}$. Men om $u\in C^2$, så är dess blandande andraderivator lika, dvs $\partial^2_{xy}u=\partial^2_{yx}u$.

$\frac{\partial}{\partial y}P=\frac{\partial}{\partial y}(\frac{\partial u}{\partial x})=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}(\frac{\partial u}{\partial y})=\frac{\partial}{\partial x}Q$ V.S.V.

b) $D=\{(x,y): 2x^2+y^2\le 1,\ x^2+y^2\ge\varepsilon^2,\ \varepsilon<\frac{1}{2}\}$.

$\omega=\frac{x+xy^2+x^3}{x^2+y^2}dx+\frac{y-x^2y-y^3}{x^2+y^2}dy \Rightarrow d\omega=0$ (exakt)



$\oint_{\partial D}\omega=\int_\gamma\omega+\int_\sigma\omega=\iint_D d\omega=0 \Leftrightarrow \int_\gamma\omega=-\int_\sigma\omega=\int_{-\sigma}\omega$

$\omega=\frac{xdx+ydy}{x^2+y^2}+\frac{(xy^2+x^3)dx-(x^2y+y^3)dy}{x^2+y^2}=\frac{1}{2}\frac{d(x^2+y^2)}{x^2+y^2}+$

$+\frac{(x^2+y^2)xdx-(x^2+y^2)ydy}{x^2+y^2}=d\ln\sqrt{x^2+y^2}+d\frac{1}{2}(x^2+y^2)$

s.a. integrationen av ω kring σ ger 0.

Funktionen U som omtalas i a) är här

$$U(x,y) = \ln\sqrt{x^2+y^2} + \tfrac{1}{2}(x^2-y^2).$$

Övning 9.21 (s. 156)

Låt γ vara randen till ett enkelt sammanhängande område i planet.

$$\int_\gamma \omega = \int_{\partial D} y^3dx + (3x-x^3)dy = \iint_D (3-3x^2-3y^2)\,dxdy = $$
$$= 3\iint_D (1-x^2-y^2)\,dxdy.$$

Denna blir så stor som möjligt för icke-negativ integrand, dvs. $1-x^2-y^2 \geq 0 \Leftrightarrow x^2+y^2 \leq 1$.

Cirkeln $\partial D = \{(x,y) : x^2+y^2 = 1\}$ är denna γ.

Anm. Läs även det som står i facit.

Övning 9.22 (s. 157)

$$\Gamma = \partial D \Rightarrow \oint_\Gamma (4y^3+y^2x-4y)dx + (8x+x^2y-x^3)dy =$$
$$= \iint_D (8+2xy-3x^2-12y^2-2xy-4)\,dxdy = \iint_D (4-3x^2-12y^2)\,dA$$
$$\Leftrightarrow 4-3x^2-12y^2 \geq 0 \Leftrightarrow 12y^2+3x^2 \leq 4 \Leftrightarrow 4y^2+\tfrac{3}{4}x^2 \leq 1.$$

Svar: Ellipskurvar $\frac{3}{4}x^2+4y^2 = 1$.

Tillämpningar på Greens formel

Övning 9.23 (s. 157)

a) $x(0) = x(1) = 0$, $y(0) = y(1) = 0$; kurvan är sluten.

$$\begin{cases} x(t) = t(1-t) \Rightarrow \dot{x}(t) = 1-2t \\ y(t) = t^2(1-t) \Rightarrow \dot{y}(t) = 2t-3t^2 \end{cases};$$

Vertikala tangenter: $\dot{x}(t) = 0 \Rightarrow t = \frac{1}{2} \Rightarrow (x,y) = (\frac{1}{4}, \frac{1}{8})$.

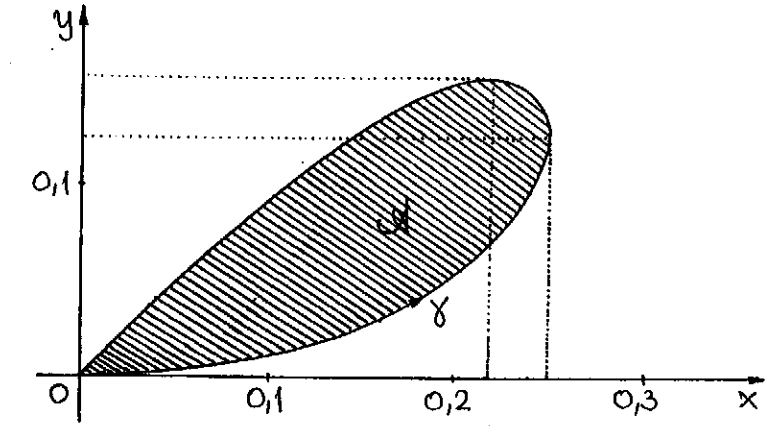
Horisontella tangenter: $\dot{y}(t) = 0 \Rightarrow t = 0 \vee t = \frac{2}{3} \Rightarrow$

$$\Rightarrow (x,y) = (0,0) \vee (x,y) = (\tfrac{2}{3}, \tfrac{4}{27}).$$

t		1/2	
$x'(t)$	+	0	−
$x(t)$	↗	1/4	↘

t		2/3	
$y'(t)$	+	0	−
$y(t)$	↘	4/27	↗

t	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
x	0	0,09	0,16	0,21	0,24	0,25	0,24	0,21	0,16	0,09
y	0	0,009	0,032	0,063	0,096	0,125	0,144	0,147	0,128	0,081

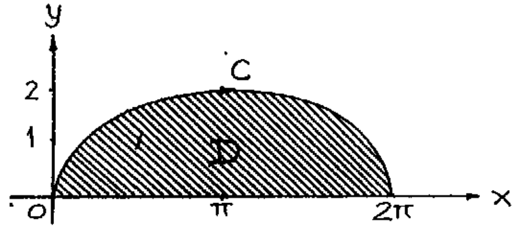


b) $\mu(\mathcal{A}) = -\oint_\gamma y(t)\dot{x}(t)\,dt = \int_0^1 (t^3-t^2)(1-2t)\,dt =$

$= \int_0^1 (3t^3-t^2-2t^4)\,dt = \left[\frac{3}{4}t^4-\frac{1}{3}t^3-\frac{2}{5}t^5\right]_0^1 =$

$= \frac{3}{4}-\frac{1}{3}-\frac{2}{5} = \frac{3\cdot3\cdot5-1\cdot4\cdot5-2\cdot3\cdot4}{3\cdot4\cdot5} = \frac{1}{60}$ ae.

Övning 9.24 (s. 157)



$\mu(D) = -\int_C x(t)\dot{y}(t)\,dt = \int_0^{2\pi} -(t-\sin t)\sin t\,dt =$

$= -\int_0^{2\pi}(t\sin t-\sin^2 t)\,dt = \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2t - t\sin t\right)dt$

$= \pi - \int_0^{2\pi} t\sin t\,dt = \pi + \left[t\cos t\right]_0^{2\pi} - \int_0^{2\pi}\cos t\,dt =$

$= \pi + 2\pi = 3\pi \approx 9{,}425$ ae.

Anm. Jag har satt ett minustecknet framför integraltecknet därför att riktningen hos C är negativ (medurs).

Övning 9.25 (s. 157)

$x^{2/3}+y^{2/3} = \cos^2 t+\sin^2 t = 1$; kurvan är en asteroid. Denna är spegelsymmetrisk m.a.p. såväl axlarna som origo. Vi studerar således kurvan i den första kvadranten. Obs. att $|x|\le 1$ och $|y|\le 1$.

$x\ge 0 \wedge y\ge 0 \Rightarrow y^{2/3} = 1-x^{2/3} \Leftrightarrow y = (1-x^{2/3})^{3/2}$;

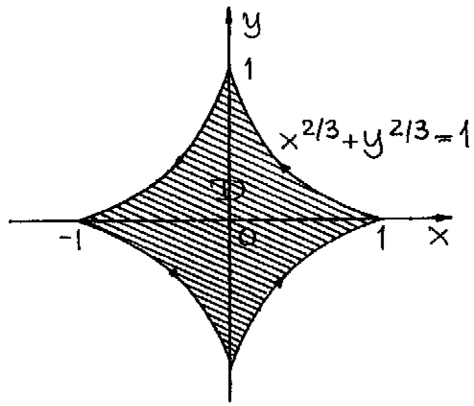
$y' = \frac{3}{2}(1-x^{2/3})^{1/2}\cdot\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1-x^{2/3})^{1/2} < 0$

$\Rightarrow$ kurvbågen är ständigt fallande.

$\begin{cases} \lim\limits_{x\to 0^+} y' = -\infty \\ \lim\limits_{x\to 1^-} y' = 0 \end{cases} \Rightarrow$ asteroiden har spetsar i punkterna $(\pm1,0)$ och $(0,\pm1)$.

Avbildningen $(x,y)\to(y,x)$ lämnar kurvan invariant $\Rightarrow$ spegelsymmetri kring $y=\pm x$.

x	0	0,1	0,2	0,3	0,35	0,4	0,5
y	1	0,69	0,53	0,41	0,35	0,31	0,23



$d\mu(D) = \frac{1}{2}(x\dot{y} - \dot{x}y)dt = \frac{1}{2}(\cos^3 t \cdot 3\sin^2 t\cos t + \sin^3 t \, 3\cos^2 t\sin t)dt$

$= \frac{1}{2} \cdot 3\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)dt = \frac{3}{2}(\sin t\cos t)^2 dt =$

$= \frac{3}{8}(2\sin t\cos t)^2 dt = \frac{3}{8}\sin^2 2t = \frac{3}{16}(1 - \cos 4t)dt \Rightarrow$

$\Rightarrow \mu(D) = \frac{3}{16}\int_0^{2\pi}(1 - \cos 4t)dt = \frac{3}{16} \cdot 2\pi = \frac{3\pi}{8} \approx 1{,}178$ ae.

Övning 9.26 (s. 157)

a) Ω är ett öppet område i $\mathbb{R}^2$.

$P(x,y)$, $Q(x,y)$ är C^1 i Ω.

$D \subset \Omega$ är kompakt och har styckvis C^1-rand γ; denna är positivt orienterad.

Då gäller

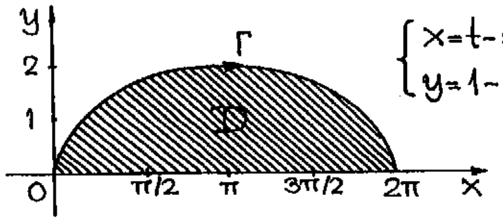
$$\int_\gamma P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

I differentialgeometrin skrivs detta

$$\int_\gamma \omega = \iint_D d\omega.$$

b) $\omega = xydy \Rightarrow Q = xy \wedge P = 0 \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$.

c)



$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$.

$\oint_{\partial D} \omega = \oint_{\partial D} xydy = -\oint_\Gamma xydy = -\int_0^{2\pi}(t - \sin t)(1 - \cos t)\sin t \, dt$

$= \left[-(t - \sin t)\frac{(1-\cos t)^2}{2}\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi}(1 - \cos t)^3 dt =$

$= \frac{1}{2}\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)dt = \frac{1}{2}\int_0^{2\pi}(1 + 3\cos^2 t)dt -$

$- \frac{1}{2}\int_0^{2\pi}\underbrace{(3\cos t + \cos^3 t)}_{\text{period } 2\pi}dt = \frac{1}{2}\int_0^{2\pi}\left(\frac{5}{2} + \frac{3}{2}\cos 2t\right)dt = \frac{5\pi}{2}$.

Anm. $3\cos t + \cos^3 t = 3\cos t + \frac{13}{4}\cos t + \frac{1}{4}\cos 3t =$

$= \frac{15}{4}\cos t + \frac{1}{4}\cos 3t \Rightarrow$ Perioden $= 2\pi$. $\Rightarrow$

$\Rightarrow \int_0^{2\pi}\left(\frac{13}{3}\cos t + \frac{1}{4}\cos 3t\right)dt = 0$.

Övning 9.27 (s. 158)

$\frac{1}{3}\oint_\gamma -y^3 dx + x^3 dy \overset{(*)}{=} \iint_D (x^2 + y^2)dxdy = I_z$.

I (∗) har jag använt Greens formel.

Övning 9.28 (s. 158)

$\oint_\gamma -ff'_y dx + ff'_x dy = \iint_D \left(\frac{\partial}{\partial x}ff'_x + \frac{\partial}{\partial y}ff'_y\right)dxdy =$

$= \iint_D (f'^2_x + ff''_{xx} + f'^2_y + ff''_{yy})dxdy =$

$= \iint_D (f(f''_{xx} + f''_{yy}) + f'^2_x + f'^2_y)dxdy =$

$= \iint_D (f'^2_x + f'^2_y)dxdy = \iint_D |\text{grad} f|^2 dxdy \overset{(*)}{=} 0 \Leftrightarrow$

$\Leftrightarrow |\text{grad} f| = 0 \Leftrightarrow \text{grad} f = 0 \Leftrightarrow f(x) = C = f(\gamma) \equiv 0$.

forts.

Anm. Integralen $\oint_\gamma -ff'_y\,dx + f\,f'_y\,dy \overset{(*)}{=} 0$, ty f är ju 0 på γ.

Potentialer och exakta differentialformer

Övning 9.29 (S. 158)

$\mathbb{F}(x,y) = (2xy,\ x^2-y^2) = (P(x,y), Q(x,y))$.

$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}2xy = 2x = \frac{\partial}{\partial x}(x^2-y^2) = \frac{\partial Q}{\partial x}$.

$\operatorname{grad} U(x,y) = (2xy,\ x^2-y^2) \Leftrightarrow \frac{\partial U}{\partial x} = 2xy \wedge \frac{\partial U}{\partial y} = x^2-y^2 \quad (*)$.

$\frac{\partial U}{\partial x} = 2xy \Leftrightarrow U(x,y) = x^2y + f(y) \Rightarrow \frac{\partial U}{\partial y} = x^2 + f'(y) \overset{(*)}{=}$

$= x^2-y^2 \Leftrightarrow f'(y) = -y^2 \Leftrightarrow f(y) = -\frac{1}{3}y^3 + C$;

Resultat: $U(x,y) = x^2y - \frac{1}{3}y^3 \quad (C = U(0,0) = 0)$.

Övning 9.30 (S. 159)

$$\begin{cases} P(x,y) = x^3-3xy^2 \Rightarrow \frac{\partial P}{\partial y} = -6xy \\ Q(x,y) = y^3-3x^2y \Rightarrow \frac{\partial Q}{\partial x} = -6xy \end{cases} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$$

$\Rightarrow \mathbb{F}(x,y) = (P(x,y), Q(x,y))$ konservativt fält.

Låt $U(x,y)$ vara potentialfunktionen

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \Rightarrow \begin{cases} \frac{\partial U}{\partial x} = P(x,y) = x^3-3xy^2 \\ \frac{\partial U}{\partial y} = Q(x,y = y^3-3x^2y \end{cases} . \quad (*)$$

$\frac{\partial U}{\partial x} = x^3-3xy^2 \Leftrightarrow U(x,y) = \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + f(y) \Rightarrow \frac{\partial U}{\partial y} =$

$= \frac{\partial}{\partial y}\left(\frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + f(y)\right) = -3x^2y + f'(y) \overset{(*)}{=} y^3 - 3x^2y$

$\Leftrightarrow f'(y) = y^3 \Leftrightarrow f(y) = \frac{1}{4}y^4 + C$;

Resultat: $U(x,y) = \frac{1}{4}x^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4$.

Anm. $dU = P(x,y)dx + Q(x,y)dy =$

$= (x^3-3xy^2)dx + (y^3-3x^2y)dy =$

$= x^3dx - (3xy^2dx + 3x^2y\,dy) + y^3dy =$

$= d\left(\frac{1}{4}x^4\right) - d\left(\frac{3}{2}x^2y^2\right) + d\left(\frac{1}{4}y^4\right) =$

$= d\left(\frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4\right) \Leftrightarrow$

$\Leftrightarrow U(x,y) = \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + C$.

Övning 9.31 (S. 159)

$\mathbb{F}(x,y) = (P(x,y), Q(x,y)) = \left(\frac{x^2-y^2}{(x^2+y^2)^2}, \frac{2xy}{(x^2+y^2)^2}\right) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} P(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow \frac{\partial P}{\partial y} = \frac{2y^3-6x^2y}{(x^2+y^2)^3} \\ Q(x,y) = \frac{2xy}{(x^2+y^2)^2} \Rightarrow \frac{\partial Q}{\partial x} = \frac{2y^3-6x^2y}{(x^2+y^2)^2} \end{cases} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};$$

$$\operatorname{grad} U(x,y) = (P(x,y), Q(x,y)) \Leftrightarrow \begin{cases} \frac{\partial U}{\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \\ \frac{\partial U}{\partial y} = \frac{2xy}{(x^2+y^2)^2} \end{cases} ; \quad (*)$$

$\frac{\partial U}{\partial y} = \frac{2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) \Leftrightarrow U(x,y) = -\frac{x}{x^2+y^2} + f(x) \Rightarrow$

$$\Rightarrow \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2} + f(x)\right) = -\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + f'(x) = \frac{x^2-y^2}{(x^2+y^2)^2} +$$

$$+ f'(x) \overset{(*)}{=} \frac{x^2-y^2}{(x^2+y^2)^2} \Leftrightarrow f'(x)=0 \Leftrightarrow f(x)=C \Rightarrow U(x,y) = -\frac{x}{x^2+y^2}.$$

Övning 9.32 (s.159)

$$\omega = g(x)(\cos xy - y\sin xy)dx - g(x)x\sin xy\,dy \text{ exakt}$$

$$\Rightarrow \frac{\partial}{\partial y} g(x)(\cos xy - y \sin xy) = \frac{\partial}{\partial x}(-g(x)x\sin xy) \Leftrightarrow$$

$$\Leftrightarrow g(x)(-x\sin xy - \sin xy - xy\cos xy) = -g'(x)x\sin xy -$$

$$-g(x)\sin xy - xyg(x)\cos xy \Leftrightarrow -xg'(x)\sin xy =$$

$$= -xg(x)\sin xy \Leftrightarrow g'(x) = g(x) \Leftrightarrow \underline{g(x) = Ce^x} \quad (C=1)$$

$$\omega = e^x(\cos xy - y\sin xy)dx - xe^x \sin xy \quad \text{exakt};$$

$$\operatorname{grad} U(x) = (P(x), Q(x)) \Rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = e^x(\cos xy - y\sin xy) \\ \dfrac{\partial U}{\partial y} = -xe^x\sin xy \end{cases} \quad (*)$$

$$\frac{\partial U}{\partial y} = -xe^x\sin xy = \frac{\partial}{\partial y}(e^x\cos xy) \Leftrightarrow U(x,y) = e^x\cos xy +$$

$$+ f(x) \Rightarrow \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(e^x\cos xy + f(x)) = e^x(\cos xy -$$

$$- y\sin xy) + f'(x) \overset{(*)}{=} e^x(\cos xy - y\sin xy) \Leftrightarrow f'(x) = 0 \Leftrightarrow$$

$$\Leftrightarrow f(x) = C \quad \Rightarrow U(x,y) = e^x\cos xy + C.$$

Resultat: $g(x) = e^x$ är en "integrerande faktor".

En potentialfunktion är $U(x,y) = e^x\cos xy$.

Övning 9.33 (s.159)

$$\omega = g(y)2xy\,dx - g(y)(y^2+3a^2-3x^2)dx \quad \text{exakt} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial}{\partial y} g(y)2xy = \frac{\partial}{\partial x}(3x^2-y^2-3a^2)g(y) \Leftrightarrow 6xg(y) =$$

$$= 2xyg'(y) + 2xg(y) \Leftrightarrow 2xyg'(y) = 4xg(y) \Leftrightarrow$$

$$\Leftrightarrow yg'(y) = 2g(y) \Leftrightarrow \frac{g'(y)}{g(y)} = \frac{2}{y} \Leftrightarrow \ln g(y) = \ln Cy^2 \Leftrightarrow$$

$$\Leftrightarrow \underline{g(y) = Cy^2} \quad (C=1).$$

$$\omega = 2xy^3dx - (y^4+3a^2y^2-3x^2y^2)dy =$$

$$= 2xy^3dx + 3x^2y^2dy - (y^4+3a^2y^2)dy =$$

$$= d(x^2y^3) - d\left(\tfrac{1}{5}y^5 + a^2y^3\right) = d\left(x^2y^3 - \tfrac{1}{5}y^5 - a^2y^3\right).$$

Svar: $g(y) = y^2$; $U(x,y) = x^2y^3 - \frac{1}{5}y^5 - a^2y^3$.

Övning 9.34 (s.159)

$$\omega = g(x,y)(x\,dx + y\,dy) \equiv x\cdot g(x,y)dx + yg(x,y)dy;$$

$$\omega \text{ exakt} \Rightarrow \frac{\partial}{\partial y} xg(x,y) = \frac{\partial}{\partial x} yg(x,y) \Leftrightarrow x\frac{\partial g}{\partial y} = y\frac{\partial g}{\partial x} \quad (*)$$

$$\Rightarrow \begin{cases} \dfrac{\partial g}{\partial x} = cx \\ \dfrac{\partial g}{\partial y} = cy \end{cases} \Leftrightarrow \begin{cases} g(x,y) = \frac{1}{2}cx^2 + \phi(y) \\ g(x,y) = \frac{1}{2}cy^2 + \psi(x) \end{cases} \Rightarrow g(x,y) = \frac{1}{2}(x^2+y^2)$$

Anm. $g(x,y) = f(x^2+y^2)$, $f \in C^1$; (*)

Resultat: $g(x,y) = \frac{1}{2}(x^2+y^2)$ till exempel.

Övning 9.35 (s. 159)

a) $\begin{cases} P(x,y) = -\dfrac{y}{x^2+y^2} \Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2} \\ Q(x,y) = \dfrac{x}{x^2+y^2} \Rightarrow \dfrac{\partial Q}{\partial x} = \dfrac{y^2-x^2}{(x^2+y^2)^2} \end{cases} \Rightarrow Pdx+Qdy$ exakt.

b) $\begin{cases} P(\cos\theta, \sin\theta) = -\sin\theta \\ Q(\cos\theta, \sin\theta) = \cos\theta \end{cases} \Rightarrow \omega = Pdx+Qdy = d\theta \Rightarrow$

$\Rightarrow \oint_{|x|=1} Pdx+Qdy = \int_0^{2\pi} d\theta = 2\pi.$

c) Villkoret $P'_y = Q'_x$ är visserligen uppfyllt men det räcker inte för att säkerställa konservatism. $Pdx+Qdy$ är inte exakt i ett område som har origo till (sin) inre punkt. Om det vore det så skulle det finnas potentialfunktion $U(x,y)$, vilket skulle ge $U(1,0)-U(1,0)=0 \neq 2\pi$ som i b) ovan.

Resultat: a) Se ovan, b) 2π. c) Nej.

Övning 9.36 (s. 159)

a) $\mathbf{F}(x,y) = (P(x,y), Q(x,y)) \Rightarrow \begin{cases} P(x,y) = -\dfrac{y}{x^2+y^2} \\ Q(x,y) = \dfrac{x}{x^2+y^2} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} P(\cos\theta, \sin\theta) = -\sin\theta \\ Q(\cos\theta, \sin\theta) = \cos\theta \end{cases} \Rightarrow \mathbf{F}\cdot d\mathbf{r} = Pdx+Qdy =$

$= (-\sin\theta)(-\sin\theta)d\theta + \cos\theta \cdot \cos\theta\, d\theta = d\theta \Rightarrow \oint_{|\mathbf{r}|=1} \mathbf{F}\cdot d\mathbf{r} =$

$= \int_0^{2\pi} d\theta = 2\pi \neq 0 \Rightarrow (P,Q)$ ej konservativ.

b) $\mathbf{F}(x,y) = (P(x,y), Q(x,y)) \Leftrightarrow \begin{cases} P(x,y) = \dfrac{x}{x^2+y^2} \\ Q(x,y) = \dfrac{y}{x^2+y^2} \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} P(\cos\theta, \sin\theta) = \cos\theta \\ Q(\cos\theta, \sin\theta) = \sin\theta \end{cases} \Rightarrow \mathbf{F}\cdot d\mathbf{r} = Pdx+Qdy =$

$= \cos\theta(-\sin\theta)d\theta + \sin\theta\cos\theta\, d\theta = 0 \Rightarrow \oint_{|\mathbf{r}|=1} \mathbf{F}\cdot d\mathbf{r} = 0. \Rightarrow$

$\Rightarrow (P,Q)$ konservativ.

Anm. Teorin finns på s. 310–311.

Övning 9.37 (s. 160)

a) Om ω är exakt, så har den en potentialfunktion $U(x,y)$ i hela området, så för en sluten kurva γ har vi $\oint_\gamma \omega = \int_a^a \operatorname{grad} U \cdot d\mathbf{r} = \int_a^a dU = [U(\mathbf{r})]_a^a = U(a)-U(a) = 0.$

b) $\omega = \dfrac{x-y}{x^2+y^2}dx + \dfrac{x+y}{x^2+y^2}dy \Rightarrow \forall x \in D: d\omega = 0$, ty

$$\frac{\partial}{\partial x}\frac{x+y}{x^2+y^2} - \frac{\partial}{\partial y}\frac{x-y}{x^2+y^2} = \frac{x^2+y^2-2x(x+y)}{(x^2+y^2)^2} - \frac{-(x^2+y^2)-2y(x-y)}{(x^2+y^2)^2} = 0.$$

D är enkelt sammanhängande så ω är exakt. $U(x,y) = \ln\sqrt{x^2+y^2} + \arctan\frac{y}{x}$ P.F.

Övning 9.38 (s. 160)

$$\omega = Pdx + Qdy \Rightarrow d\omega = \left(\frac{2y}{(1+x^2y^2)^2} - \frac{2y}{(1+x^2y^2)^2}\right)dxdy = 0$$

$\Rightarrow \omega$ exakt $\Rightarrow U$ finns för alla (x,y).

$$\operatorname{grad} U(x,y) = (P(x,y), Q(x,y)) \Rightarrow \begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{y^2}{1+x^2y^2} \\[2mm] \dfrac{\partial U}{\partial y} = \dfrac{xy}{1+x^2y^2} + \arctan xy \end{cases} (*)$$

$$\frac{\partial U}{\partial x} = \frac{y^2}{1+x^2y^2} \Rightarrow U(x,y) = \int \frac{y^2}{1+x^2y^2}dx \left[\begin{matrix} v = xy \Leftrightarrow x = v/y \\ dx = dv/y \end{matrix}\right] =$$

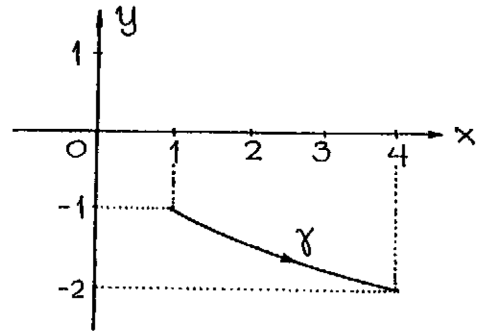
$$= \left\{\int y \frac{1}{1+v^2}dv\right\}_{v=xy} = y\cdot\arctan(xy) + f(y) \Rightarrow$$

$$\Rightarrow \frac{\partial U}{\partial y} = \arctan(xy) + \frac{xy}{1+x^2y^2} + f'(y) \overset{(*)}{=} \frac{xy}{1+x^2y^2} + \arctan xy$$

$$\Leftrightarrow f'(y) = 0 \Leftrightarrow f(y) = C. \Rightarrow \underline{U(x,y) = y\arctan xy + C.}$$

$$\int_\gamma \omega = \int_{(0,0)}^{(1,1)} d(y\arctan xy) = [y\cdot\arctan(xy)]_{(0,0)}^{(1,1)} = \frac{\pi}{4}.$$

Övning 9.39 (s. 160)



I den fjärde kvadranten är $\omega = Pdx + Qdy$ av klass C^1. Vi undersöker exakthet. forts.

$$\omega = \left(\sqrt{x^2-y} + \frac{x^2}{\sqrt{x^2-y}}\right)dx - \frac{x}{2\sqrt{x^2-y}}dy = \frac{2x^2-y}{\sqrt{x^2-y}}dx - \frac{x}{2\sqrt{x^2-y}}dy$$

$$d\omega = \left(\frac{\partial}{\partial x}\frac{-x}{\sqrt{x^2-y}} - \frac{\partial}{\partial y}\frac{2x^2-y}{\sqrt{x^2-y}}\right)dxdy =$$

$$= \left(\frac{\partial}{\partial x}(-x(x^2-y)^{-1/2}) - \frac{\partial}{\partial y}(2x^2-y)(x^2-y)^{-1/2}\right)dxdy =$$

$$= \left(\frac{y}{(x^2-y)^{3/2}} - \frac{y}{(x^2-y)^{3/2}}\right)dxdy = 0 \Rightarrow \omega \text{ exakt} \Rightarrow$$

$\Rightarrow$ potentialfunktion finns.

$$\operatorname{grad} U(x,y) = (P(x,y), Q(x,y)) \Leftrightarrow \begin{cases} P(x,y) = \dfrac{2x^2-y}{\sqrt{x^2-y}} \\[2mm] Q(x,y) = \dfrac{-x}{2\sqrt{x^2-y}} \end{cases} ; (*)$$

$$\frac{\partial U}{\partial y} = -\frac{x}{2\sqrt{x^2-y}} = \frac{\partial}{\partial y} x\sqrt{x^2-y} \Leftrightarrow U(x,y) = x\sqrt{x^2-y} + f(x)$$

$$\Rightarrow \frac{\partial U}{\partial x} = \sqrt{x^2-y} + \frac{x^2}{\sqrt{x^2-y}} + f'(x) \overset{(*)}{=} \sqrt{x^2-y} + \frac{x^2}{\sqrt{x^2-y}} \Leftrightarrow f'(x) = 0$$

$$\Leftrightarrow f(x) = C \Rightarrow U(x,y) = x\sqrt{x^2-y} + C.$$

$$\int_\gamma \omega = \int_{(1,-1)}^{(4,-2)} d(x\sqrt{x^2-y}) = \left[x\sqrt{x^2-y}\right]_{(1,-1)}^{(4,-2)} = 4\sqrt{18} - \sqrt{2} =$$

$$= 4\cdot 3\sqrt{2} - \sqrt{2} = 12\sqrt{2} - \sqrt{2} = \underline{11\sqrt{2}.}$$

Övning 9.40 (s. 160)

$$\omega = 2xye^{x^2+y}dx + (1+y)e^{x^2+y}dy ;$$

$$d\omega = \left(\frac{\partial}{\partial x}(1+y)e^{x^2+y} - \frac{\partial}{\partial y}2xye^{x^2+y}\right)dxdy =$$

$$= \left((1+y)2x e^{x^2+y} - (2x+2xy)e^{x^2+y}\right)dxdy =$$

$$= e^{x^2+y}\left((1+y)\cdot 2x - (2x+2xy)\right)dxdy = 0 \Rightarrow \omega \text{ exakt}.$$

$$\operatorname{grad} U(x,y) = (P(x,y), Q(x,y)) \Leftrightarrow \begin{cases} \frac{\partial U}{\partial x} = 2xye^{x^2+y} \\ \frac{\partial U}{\partial y} = (1+y)e^{x^2+y} \end{cases}; \ (*)$$

$$\frac{\partial U}{\partial x} = 2xye^{x^2+y} = \frac{\partial}{\partial x} ye^{x^2+y} \Leftrightarrow U(x,y) = ye^{x^2+y} + f(y)$$

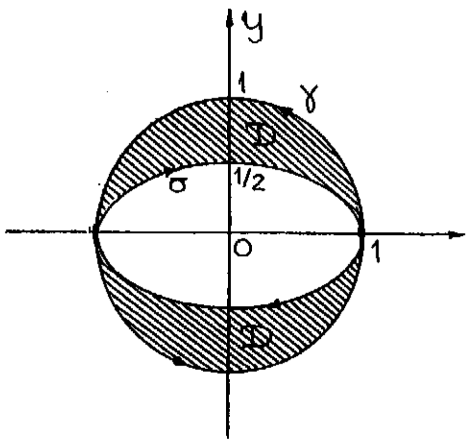
$$\Rightarrow \frac{\partial U}{\partial y} = e^{x^2+y} + ye^{x^2+y} + f'(y) \overset{(*)}{=} (1+y)e^{x^2+y} \Leftrightarrow f'(y) = 0$$

$$\Leftrightarrow f(y) = C \Rightarrow U(x,y) = ye^{x^2+y} + C;$$

$$\int_\gamma \omega = \int_{(0,0)}^{(1,1)} d(ye^{x^2+y}) = [ye^{x^2+y}]_{(0,0)}^{(1,1)} = e^2.$$

Blandade problem

Övning 9.41 (s. 160)



$$D = \{(x,y) : x^2+y^2 \leq 1 \ \wedge \ x^2+4y^2 \geq 1\}.$$

$$\mathbb{F}(x,y) = \left(-\frac{y}{x^2+4y^2}, \frac{x}{x^2+4y^2}\right);$$

$$\omega = \mathbb{F}\cdot d\mathbf{r} = \left(-\frac{y}{x^2+4y^2}, \frac{x}{x^2+4y^2}\right)\cdot(dx,dy) = \frac{-ydx+xdy}{x^2+4y^2};$$

$$\forall x \in D: \ d\omega = \left(\frac{\partial}{\partial x}\frac{x}{x^2+4y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+4y^2}\right)dxdy =$$

$$= \left(\frac{x^2+4y^2-x\cdot 2x}{(x^2+4y^2)^2} + \frac{x^2+4y^2-y\cdot 8y}{(x^2+4y^2)^2}\right)dxdy =$$

$$= \frac{x^2+4y^2-2x^2+x^2+4y^2-8y^2}{(x^2+4y^2)^2}dxdy = 0 \Rightarrow \omega \text{ exakt.}$$

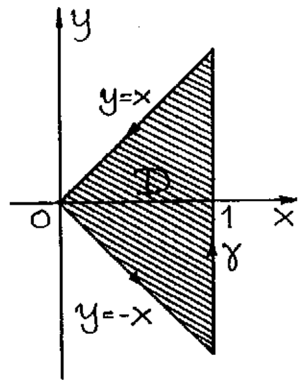
$$\oint_{\partial D}\omega = \iint_D d\omega = 0 \Leftrightarrow \int_\gamma \omega + \int_\sigma \omega = 0 \Leftrightarrow \int_\gamma \omega = -\int_\sigma \omega =$$

$$= \int_{-\sigma}\omega \overset{(*)}{=} \int_0^{2\pi}\left(-\tfrac{1}{2}\sin\theta(-\sin\theta) + \cos\theta\cdot\tfrac{1}{2}\cos\theta\right)d\theta = \frac{1}{2}\int_0^{2\pi}d\theta = \pi.$$

Anm. I (*) har jag infört $\sigma(\theta) = (\cos\theta, \frac{1}{2}\sin\theta)$.

$\oint_{\partial D}\omega = \iint_D d\omega$ är Greens formel.

Övning 8.42 (s. 160)



$$\omega = (ye^{x^2} + x\sin(x^2+y^2))dx + ((1+xy)^2 + y\sin(x^2+y^2))dy$$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy = (2y + 2xy^2 - e^{x^2})dxdy;$$

$$\oint_\gamma \omega = \iint_D d\omega = \iint_D (2y+2xy^2-e^{x^2})dxdy =$$

$$= \int_0^1\left(\int_{-x}^{x}(2y+2xy^2-e^{x^2})dy\right)dx = \qquad \text{forts.}$$

$$= \int_0^1 \left(\left[\tfrac{2}{3}xy^3 + y^2 - ye^{x^2} \right]_{y=-x}^{y=x} \right) dx =$$

$$= \int_0^1 \left(\tfrac{2}{3}x^4 + x^2 - xe^{x^2} + \tfrac{2}{3}x^4 - x^2 - xe^{x^2} \right) dx =$$

$$= \int_0^1 \left(\tfrac{4}{3}x^4 - 2xe^{x^2} \right) dx = \left[\tfrac{4}{15}x^5 - e^{x^2} \right]_0^1 = \tfrac{4}{15} - e + 1 = \tfrac{19}{15} - e.$$

Övning 9.43 (s. 161)

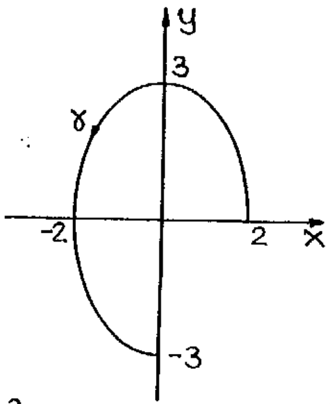
a) $\omega = \mathbb{F}\cdot d\mathbf{r} = f(r)x\,dx + f(r)y\,dy \Rightarrow \begin{cases} P(x,y) = f(r)x \\ Q(x,y) = f(r)y \end{cases}$ (*)

$U(x,y) = \int_0^r t f(t)\,dt$;

$$\begin{cases} \dfrac{\partial U}{\partial x} = \dfrac{\partial}{\partial x}\displaystyle\int_0^r t f(t)\,dt = \dfrac{d}{dr}\displaystyle\int_0^r t f(t)\,dt \cdot \dfrac{\partial r}{\partial x} = r f(r)\cdot \dfrac{x}{r} = f(r)x. \\ \dfrac{\partial U}{\partial y} = \dfrac{\partial}{\partial y}\displaystyle\int_0^r t f(t)\,dt = \dfrac{d}{dr}\displaystyle\int_0^r t f(t)\,dt \cdot \dfrac{\partial r}{\partial y} = r f(r)\cdot \dfrac{y}{r} = f(r)y. \end{cases}$$

$\Rightarrow \operatorname{grad} U(x,y) = (f(r)x, f(r)y) = f(r)(x,y) = f(r)\cdot\mathbf{r}.$

b)



$$\omega = (x^3 + xy^2)e^{x^2+y^2}dx + (x^2y + y^3)e^{x^2+y^2}dy =$$

$$= (x^2+y^2)e^{x^2+y^2}x\,dx + (x^2+y^2)e^{x^2+y^2}y\,dy =$$

$$= r^2e^{r^2}(x,y)(dx,dy) \Rightarrow \mathbb{F}(x,y) = r^2e^{r^2}(x,y) \Rightarrow$$

$$\Rightarrow f(r) = r^2e^{r^2} \Rightarrow U(x,y) = \int_0^r t^3e^{t^2}dt \Rightarrow$$

$$\Rightarrow U(0,-3) - U(2,0) = \int_0^3 t^3e^{t^2}dt - \int_0^2 t^3e^{t^2}dt =$$

$$= \int_2^3 t^3e^{t^2}dt \left[\begin{matrix} v = t^2 \\ dv = 2t\,dt \end{matrix} \;\middle|\; \begin{matrix} 3\to 9 \\ 2\to 4 \end{matrix} \right] = \int_4^9 \tfrac{1}{2}ve^v dv =$$

$$= \left[\tfrac{1}{2}(v-1)e^v \right]_4^9 = \tfrac{1}{2}(8e^9 - 3e^4) = 4e^9 - \tfrac{3}{2}e^4.$$

Övning 9.44 (s. 161)

(i) $\mathbb{F} = (h,-g) \wedge \dfrac{\partial h}{\partial y} + \dfrac{\partial g}{\partial x} = 0 \Rightarrow \mathbb{F}$ konservativt $\Rightarrow$

$\Rightarrow \exists U \in C^2 :\ \operatorname{grad} U(x,y) = (h,-g).$

Anm. $\mathbb{F} = (P,Q)$ konservativt $\Rightarrow \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(ii) $\mathbb{G} = (-g,f) \wedge \dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} = 0 \Rightarrow \mathbb{G}$ konservativ $\Rightarrow$

$\Rightarrow \exists V \in C^2 :\ \operatorname{grad} V(x,y) = (-g,f).$

(iii) $\operatorname{grad} U = (h,-g) \Rightarrow U(x,y) = \int_\gamma h\,dx - g\,dy$

$\operatorname{grad} V = (-g,f) \Rightarrow V(x,y) = \int_\delta -g\,dx + f\,dy$ (*)

(iv) $\mathbb{H} = (U,V) \overset{(*)}{\Rightarrow} \dfrac{\partial U}{\partial y} = -g = \dfrac{\partial V}{\partial x} \Rightarrow \mathbb{H}$ konservativt

$\Rightarrow \exists \phi \in C^2 :\ \operatorname{grad}\phi(x,y) = (U,V)$;

(v) $\dfrac{\partial \phi}{\partial x} = U \overset{(*)}{\Rightarrow} \dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial U}{\partial x} = h$;

$\dfrac{\partial \phi}{\partial y} = V \overset{(**)}{\Rightarrow} \dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{\partial V}{\partial x} = -g \wedge \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{\partial V}{\partial y} = f.$

forts.

Anm. Om du inte har förstått (iv) och (v) bör du gå igenom Sats 3 på s. 308 i grundboken.

Övning 9.45 (s. 161)

$$\omega = P\,dx + Q\,dy \Rightarrow \begin{cases} P(x,y) = \dfrac{2x}{2x^2+3y^2} \\ Q(x,y) = \dfrac{3y}{2x^2+3y^2} \end{cases} \Rightarrow \frac{\partial P}{\partial y} = -\frac{12xy}{(2x^2+3y^2)^2} = \frac{\partial Q}{\partial x}$$

$\Rightarrow \omega$ exakt $\Rightarrow \exists\, U \in C^2$: $\operatorname{grad} U = (P, Q)$; $(*)$

$$\frac{\partial U}{\partial x} = \frac{2x}{2x^2+3y^2} = \frac{\partial}{\partial x}\,\frac{1}{2}\ln(2x^2+3y^2) = \frac{\partial}{\partial x}\ln\sqrt{2x^2+3y^2} \Leftrightarrow$$

$$\Leftrightarrow U(x,y) = \ln\sqrt{2x^2+3y^2} + f(y) \Rightarrow \frac{\partial U}{\partial y} = \frac{3y}{2x^2+3y^2} + f'(y)$$

$$\overset{(*)}{=} Q = \frac{3y}{2x^2+3y^2} \Leftrightarrow f'(y) = 0 \Leftrightarrow f(y) = C \Rightarrow U = \ln\sqrt{2x^2+3y^2}.$$

$x(0) = 2,\ y(0) = e$; $x(\pi) = 0,\ y(\pi) = \frac{1}{e}$;

$$\int_C \omega = \int_{(2,e)}^{(0,1/e)} d\left(\ln\sqrt{2x^2+3y^2}\right) = \left[\ln\sqrt{2x^2+3y^2}\right]_{(2,e)}^{(0,e^{-1})} =$$

$$= \frac{1}{2}\ln\frac{3}{e^2} - \frac{1}{2}\ln(8+3e^2) = \underline{\frac{1}{2}\ln\frac{3e^{-2}}{8+3e^2}}.$$

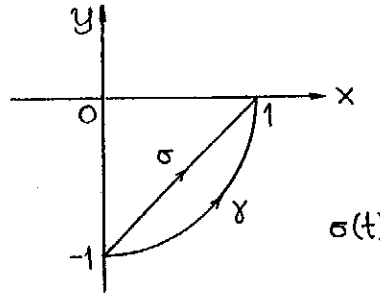
Övning 9.46 (s. 161)

γ ligger i området $\Omega = \{(x,y) : y < x\}$, s.a. $\omega \in C^1$.

$$\begin{cases} P(x,y) = -\dfrac{2y}{(x-y)^3} \Rightarrow \dfrac{\partial P}{\partial y} = -\dfrac{4y-2x}{(x-y)^4} \\ Q(x,y) = \dfrac{x+y}{(x-y)^3} \Rightarrow \dfrac{\partial Q}{\partial x} = -\dfrac{4y-2x}{(x-y)^4} \end{cases} \Rightarrow \omega \text{ exakt} \Rightarrow$$

$\Rightarrow \int_\gamma \omega$ oberoende av γ (vägen).

forts.



$\sigma(t) = (t, t-1),\ 0 \le t \le 1.$

$$\int_\gamma \omega = \int_\sigma \omega = \int_0^1 \omega(\sigma) = \int_0^1 (-2(t-1) + 2t - 1)\,dt = \int_0^1 dt = 1$$

Anm. $\omega(\sigma)$ tolkas så att man sätter in $x = t$ och $y = t-1$ i $\omega = \dfrac{-2y\,dx + (x+y)\,dy}{(x-y)^3}$.

Övning 9.47 (s.162)

$$W = f(a,b) = \oint_\gamma (ax+by)^3(dx+dy)\left[\begin{matrix} x = \cos t \\ y = \sin t \end{matrix},\ 0 \le t \le 2\pi\right] =$$

$$= \int_0^{2\pi} (a\cos t + b\sin t)^3(\cos t - \sin t)\,dt =$$

$$= \int_0^{2\pi} (a^3\cos^3 t + 3a^2b\cos^2 t\sin t + 3ab^2\cos t\sin^2 t + b^3\sin^3 t)\cos t\,dt$$

$$- \int_0^{2\pi} (a^3\cos^3 t + 3a^2b\cos^2 t\sin t + 3ab^2\cos t\sin^2 t + b^3\sin^3 t)\sin t\,dt.$$

$$= \int_0^{2\pi} (a^3\cos^4 t + 3ab(b-a)\cos^2 t\sin^2 t - b^3\sin^4 t)\,dt +$$

$$+ \int_0^{2\pi} (3a^2b\cos^3 t\sin t + 3ab^2\sin^3 t\cos t)\,dt =$$

$$= \int_0^{2\pi} (a^3\cos^4 t + \tfrac{3}{4}ab(b-a)\sin^2 2t - b^3\sin^4 t)\,dt;$$

$$\cos^4 t = (\cos^2 t)^2 = \tfrac{1}{4}(1+\cos 2t)^2 = \tfrac{1}{4}(1 + 2\cos 2t + \cos^2 2t) =$$

$$= \tfrac{1}{4}(1 + 2\cos 2t + \tfrac{1}{2} + \tfrac{1}{2}\cos 4t) = \tfrac{1}{8}(3 + 4\cos 2t + \cos 4t);$$

$$\sin^4 t = (\sin^2 t)^2 = \frac{1}{4}(1-\cos 2t)^2 = \frac{1}{4}(1-2\cos 2t+\cos^2 2t) =$$
$$= \frac{1}{4}\left(1-2\cos 2t+\frac{1}{2}+\frac{1}{2}\cos 4t\right) = \frac{1}{8}(3-4\cos 2t+\cos 4t).$$

$$f(a,b) = \int_0^{2\pi}\left(\frac{3}{8}a^3+\frac{3}{8}ab(b-a)-\frac{3}{8}b^3\right)dt =$$
$$= \frac{3\pi}{4}(a^3+ab^2-a^2b-b^3)\ ,\quad a^2+b^2\leq 1.$$

Lägg märke till att när man integrerar $\cos kx$ och $\sin kx$ över en period blir bidraget 0.

$$\frac{\partial f}{\partial a} = \frac{3\pi}{4}(3a^2+b^2-2ab)\ ,\quad \frac{\partial f}{\partial b} = \frac{3\pi}{4}(2ab-a^2-3b^2).$$

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = 0 \Rightarrow \begin{cases} 3a^2+b^2-2ab=0 \\ 3b^2+a^2-2ab=0 \end{cases} \Leftrightarrow \begin{cases} a^2=b^2 \\ 3a^2+b^2-2ab=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow a=b=0 \Rightarrow \underline{f(0,0)=0}.$$

$$f(\cos\theta,\sin\theta) = \frac{3\pi}{4}(\cos^3\theta+\cos\theta\sin^2\theta-\cos^2\theta\sin\theta-\sin^3\theta)$$
$$= \frac{3\pi}{4}(\cos^3\theta+\cos\theta-\cos^3\theta-\sin\theta+\sin^3\theta-\sin^3\theta) =$$
$$= \frac{3\pi}{4}(\cos\theta-\sin\theta) = \frac{3\pi}{4}\sqrt{2}\cos\left(\theta+\frac{\pi}{4}\right) \in \left[-\frac{3\pi\sqrt{2}}{4},\ \frac{3\pi\sqrt{2}}{4}\right].$$

<u>Resultat:</u> Mellan $-\frac{3\pi\sqrt{2}}{4}$ och $\frac{3\pi\sqrt{2}}{4}$.

Övning 9.48 (s. 162)

a) $\begin{cases} x=\sin t \\ y=t\sin t \end{cases} \Rightarrow \begin{cases} \dot{x}=\cos t \\ \dot{y}=t\cos t+\sin t \end{cases},\ 0\leq t\leq\pi.$

Lodräta tangenten fås för $\dot{x}=0$, dvs. för $t=\frac{\pi}{2}$. En sådan finns alltså i punkten $(1,\frac{\pi}{2})$.

Vågräta tangenter fås för $\dot{y}=0$, dvs. för $t=0$ och för $t=2{,}03$. Motsvarande punkter är origo $(0,0)$ och $(0{,}90;\ 1{,}82)$.

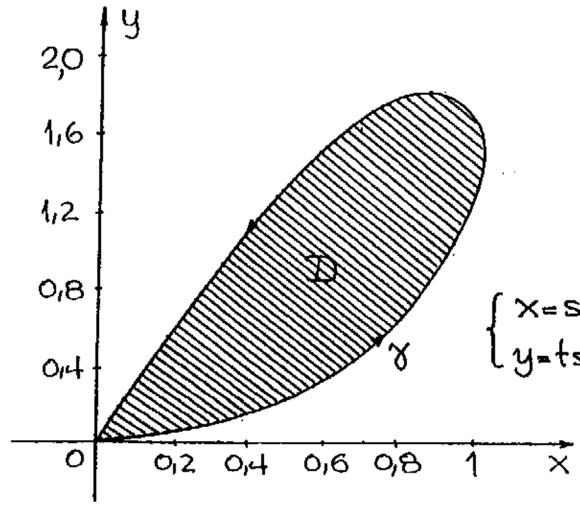
$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{t\cos t+\sin t}{\cos t} = t+\tan t = \phi(t)$$

$\phi'(t) = 1+\frac{1}{\cos^2 t} > 0 \Rightarrow$ kurvan är en ögla (sluten).

t	0		$\pi/2$		π
$\dot{x}(t)$		+	0	−	
$x(t)$		↗	$\pi/2$	↘	

t	0		2,03		π
$\dot{y}(t)$		+	0	−	
$y(t)$		↗	1,82	↘	

t	0	0,4	0,8	1,2	1,6	2,0	2,4	2,8	3,0	3,1	π
x	0	0,39	0,72	0,93	1,00	0,91	0,68	0,33	0,14	0,04	0
y	0	0,16	0,57	1,12	1,60	1,82	1,62	0,94	0,42	0,13	0



$\begin{cases} x=\sin t \\ y=t\sin t \end{cases},\ t\in[0,\pi].$

b) $\mu(D) = -\oint_\gamma y\,dx = -\int_0^\pi t\sin t\cos t\,dt = \left[\frac{t}{2}\cos^2 t\right]_0^\pi -$

$-\frac{1}{2}\int_0^\pi \cos^2 t\,dt = \frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} \approx 0{,}785$ ae.

Övning 9.49 (s. 162)

$\omega = \frac{1+x^2-y^4}{x+y^2}dx + \frac{2y}{x+y^2}dy$; $\Omega = \{(x,y) : x+y+1>0\}$.

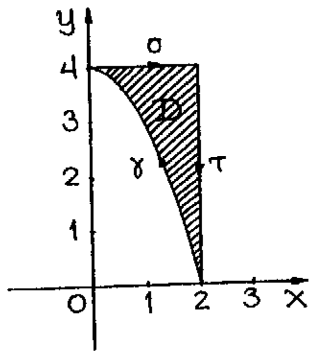
$$d\omega = \left(\frac{\partial}{\partial x}\frac{2y}{x+y^2} - \frac{\partial}{\partial y}\frac{1+x^2-y^4}{x+y^2}\right)dxdy =$$

$$= \left(\frac{-2y}{(x+y^2)^2} - \frac{-4y^3(x+y^2)-2y(1+x^2-y^4)}{(x+y^2)^2}\right)dxdy =$$

$$= \frac{\cancel{-2y}+4xy^3+4y^5+\cancel{2y}+2x^2y-2y^5}{(x+y^2)^2}dxdy =$$

$$= \frac{2y^5+4xy^3+2x^2y}{(x+y^2)^2}dxdy = \frac{2y(y^4+2xy^2+x^2)}{(x+y^2)^2}dxdy =$$

$$= 2y\,dxdy\ ;$$



$$\oint_{\partial D}\omega = \iint_D d\omega \Leftrightarrow \int_\gamma \omega + \int_\sigma \omega + \int_\tau \omega = \iint_D d\omega \Leftrightarrow$$

$$\Leftrightarrow \int_\gamma \omega = -\int_\sigma \omega - \int_\tau \omega + \iint_D d\omega = -\int_\sigma \omega + \int_{-\tau}\omega + \iint_D d\omega =$$

$$= -\int_0^2 \frac{x^2-255}{x+16}dx + \int_0^4 \frac{2y}{y^2+2}dy + \int_0^2\left(\int_{4-x^2}^4 2y\,dy\right)dx =$$

$$= -\int_0^2\left(x-16+\frac{1}{x+16}\right)dx + \ln 9 + \int_0^2 (16-(4-x^2)^2)dx = \ln 8 + \frac{226}{15}.$$